"JUST THE MATHS"

SLIDES NUMBER

10.8

DIFFERENTIATION 8
(Higher derivatives)

by

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10.8.1 The theory

UNIT 10.8 - DIFFERENTIATION 8

HIGHER DERIVATIVES

10.8.1 THE THEORY

In most examples on differentiating a function of x with respect to x, the result obtained is **another** function of x.

The possibilty arises of differentiating again with respect to x.

ILLUSTRATION

Let

$$y = f(x)$$

represent the distance, y, travelled by a moving object at time, x.

(a) The **speed** of the moving object is $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(b) The **acceleration** is defined as the rate of increase of speed with respect to time.

It is therefore represented by the symbol

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right].$$

The second derivative of y with respect to x is usually written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

and is pronounced "d two y by dx squared".

We could, if necessary, differentiate over and over again to obtain the symbols

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \text{ and } \frac{\mathrm{d}^4y}{\mathrm{d}x^4}.$$

EXAMPLES

1. If $y = \sin 2x$ show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 0.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin 2x = -4y.$$

2. If $y = x^4$, show that every derivative of y with respect to x after the fourth derivative is zero.

   Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3;$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2;$$
$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 24x;$$
$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 24.$$

   We now have a constant function so that all future derivatives will be zero.

   Note:

   In general, every derivative of $y = x^n$ after the n-th derivative will be zero.

3. If $x = 3t^2$ and $y = 6t$, obtain an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t.

   Solution

   Firstly,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

   Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}.$$

Secondly,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\mathrm{d}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x}.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

This is a general formula

In the present example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}\right]}{6t} = \frac{\frac{-1}{t^2}}{6t} = -\frac{1}{6t^3}.$$

Note:

For a function $f(x)$, an alternative notation for the derivatives of order two, three, four, etc. is

$$f''(x), \quad f'''(x), \quad f^{(iv)}(x), \text{ etc.}$$