"JUST THE MATHS"

SLIDES NUMBER

10.5

DIFFERENTIATION 5
(Implicit and parametric functions)

by

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UNIT 10.5 - DIFFERENTIATION 5
IMPLICIT & PARAMETRIC FUNCTIONS

10.5.1 IMPLICIT FUNCTIONS

Some relationships between two variables x and y do not give y explicitly in terms of x (or x explicitly in terms of y).

ILLUSTRATIONS

1.
$$x^2 + y^2 = 16$$
is not explicit for either x or y.
We could, however, write
$$y = \pm\sqrt{16 - x^2} \quad \text{or} \quad x = \pm\sqrt{16 - y^2}.$$

2. By contrast, consider
$$x^2y^3 + 9\sin(5x - 7y) = 10.$$
There is no way of stating either variable explicitly in terms of the other.

Such relationships between x and y are said to be **'implicit relationships"** and **"implicit differentiation"** means differentiate each term in the relationship with respect to the same variable without attempting to rearrange the formula.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + y^2 = 16.$$

Solution

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{2y} = -\frac{x}{y}.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + 2xy^3 + y^5 = 4.$$

Solution

$$2x + 2\left[x.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.1\right] + 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

On rearrangement,

$$\left[6xy^2 + 5y^4\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -(2x + 2y^3).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + 2y^3}{6xy^2 + 5y^4}.$$

3. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

Solution

$$x^2.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.2x + 9\cos(5x - 7y).\left[5 - 7\frac{\mathrm{d}y}{\mathrm{d}x}\right] = 0.$$

On rearrangment,

$$\left[3x^2y^2 - 63\cos(5x - 7y)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[2xy^3 + 45\cos(5x - 7y)\right].$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^3 + 45\cos(5x - 7y)}{3x^2y^2 - 63\cos(5x - 7y)}.$$

10.5.2 PARAMETRIC FUNCTIONS

Sometimes, the variables x and y can be expressed in terms of a third variable, usually t or θ, called a **"parameter"**.

In general, we write

$$x = x(t) \quad \text{and} \quad y = y(t).$$

From the Function of a Function Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x},$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}.$$

EXAMPLES

1. Determine an expression for $\frac{dy}{dx}$ in terms of t in the case when

$$x = 3t^2 \quad \text{and} \quad y = 6t.$$

Solution

$$\frac{dy}{dt} = 6 \quad \text{and} \quad \frac{dx}{dt} = 6t.$$

Hence,

$$\frac{dy}{dx} = \frac{6}{6t} = \frac{1}{t}.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of θ in the case when

$$x = \sin^3\theta \quad \text{and} \quad y = \cos^3\theta.$$

Solution

$$\frac{dx}{d\theta} = 3\sin^2\theta.\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = -3\cos^2\theta.\sin\theta.$$

Hence,

$$\frac{dy}{dx} = \frac{-3\cos^2\theta.\sin\theta}{3\sin^2\theta.\cos\theta}.$$

That is,

$$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$