“JUST THE MATHS”

SLIDES NUMBER

10.4

DIFFERENTIATION 4
(Products and quotients)
&
(Logarithmic differentiation)

by

A.J.Hobson

UNIT 10.4 - DIFFERENTIATION 4

PRODUCTS, QUOTIENTS AND LOGARITHMIC DIFFERENTIATION

10.4.1 PRODUCTS

Suppose

$$y = u(x)v(x),$$

where $u(x)$ and $v(x)$ are two functions of x.

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δu in u, δv in v and δy in y.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(u + \delta u)(v + \delta v) - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{uv + u\delta v + v\delta u + \delta u\delta v - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x}\right].$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[u.v] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Hint: Think of this as
(FIRST times DERIVATIVE OF SECOND) plus (SECOND times DERIVATIVE OF FIRST)

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = x^7 \cos 3x.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^7.-3\sin 3x+\cos 3x.7x^6 = x^6[7\cos 3x-3x\sin 3x].$$

2. Evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = -1$ in the case when

$$y = (x^2 - 8)\ln(2x + 3).$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2 - 8).\frac{1}{2x+3}.2 + \ln(2x+3).2x$$

$$= 2\left[\frac{x^2-8}{2x+3} + x\ln(2x+3)\right].$$

When $x = -1$, this has value -14 since $\ln 1 = 0$.

10.4.2 QUOTIENTS

Suppose that

$$y = \frac{u(x)}{v(x)}.$$

We may write

$$y = u(x).[v(x)]^{-1}.$$

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u.(-1)[v]^{-2}.\frac{\mathrm{d}v}{\mathrm{d}x} + v^{-1}.\frac{\mathrm{d}u}{\mathrm{d}x},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

EXAMPLES

1. Show that the derivative with respect to x of $\tan x$ is $\sec^2 x$.

   Solution

   $$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x.\cos x - \sin x. - \sin x}{\cos^2 x}$$

   $$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

2. Determine an expression for $\frac{dy}{dx}$ in the case when

$$y = \frac{2x+1}{(5x-3)^3}.$$

Solution

Using $u(x) \equiv 2x+1$ and $v(x) \equiv (5x-3)^3$,

$$\frac{dy}{dx} = \frac{(5x-3)^3.2 - (2x+1).3(5x-3)^2.5}{(5x-3)^6}.$$

That is,

$$\frac{dy}{dx} = \frac{(5x-3).2 - 15(2x+1)}{(5x-3)^4} = -\frac{20x+21}{(5x-3)^4}.$$

Note:

A modified version of the Quotient Rule is for quotients in the form

$$\frac{u}{v^n}.$$

If

$$y = \frac{u}{v^n},$$

then,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - nu\frac{dv}{dx}}{v^{n+1}}.$$

In Example 2 above, we could write

$$u \equiv 2x+1 \quad v \equiv 5x-3 \quad \text{and} \quad n = 3.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3).2 - 3(2x+1).5}{(5x-3)^4},$$

as before.

10.4.3 LOGARITHMIC DIFFERENTIATION

(a) Functions containing a variable index

First consider the **"exponential function"**, e^x.

Letting

$$y = e^x,$$

we may write

$$\ln y = x.$$

Differentiating both sides **with respect to** x, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y = e^x.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right] = e^x.$$

Notes:

(i) Differentiating $x = \ln y$ with respect to y,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{y}.$$

But it can be shown that, for most functions,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}},$$

so that the same result is obtained as before.

(ii) The derivative of e^x may easily be used to establish the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh x] = \cosh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cosh x] = \sinh x,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh x] = \mathrm{sech}^2\, x.$$

We use the definitions

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}, \quad \cosh x \equiv \frac{e^x + e^{-x}}{2},$$

and

$$\tanh x \equiv \frac{\sinh x}{\cosh x}.$$

FURTHER EXAMPLES

1. Write down the derivative with respect to x of the function
$$e^{\sin x}.$$
Solution
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin x}\right] = e^{\sin x}.\cos x.$$

2. Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when
$$y = (3x+2)^x.$$
Solution

Taking natural logarithms of both sides,
$$\ln y = x\ln(3x+2).$$
Differentiating both sides with respect to x,
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x.\frac{3}{3x+2} + \ln(3x+2).1.$$
Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x+2)^x\left[\frac{3x}{3x+2} + \ln(3x+2)\right].$$

(b) Products or Quotients with more than two elements

We illustrate with examples:

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = e^{x^2}.\cos x.(x+1)^5.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x^2 + \ln\cos x + 5\ln(x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{\sin x}{\cos x} + \frac{5}{x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2}.\cos x.(x+1)^5\left[2x - \tan x + \frac{5}{x+1}\right].$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{e^x . \sin x}{(7x+1)^4}.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x + \ln \sin x - 4\ln(7x+1).$$

Differentiating both sides with repect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{\cos x}{\sin x} - 4.\frac{7}{7x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x . \sin x}{(7x+1)^4}\left[1 + \cot x - \frac{28}{7x+1}\right].$$

Note:

In all examples on logarithmic differentiation, the original function will appear as a factor at the beginning of its derivative.