“JUST THE MATHS”

SLIDES NUMBER

10.3

DIFFERENTIATION 3
(Elementary techniques of differentiation)

by

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UNIT 10.3 - DIFFERENTIATION 3

ELEMENTARY TECHNIQUES OF DIFFERENTIATION

10.3.1 STANDARD DERIVATIVES

$f(x)$	$f'(x)$
a const.	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$

10.3.2 RULES OF DIFFERENTIATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants.

Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[Af(x)+Bg(x)\right]=A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]+B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

Proof:

The left-hand-side is equivalent to

$$\lim_{\delta x\to 0}\frac{[Af(x+\delta x)+Bg(x+\delta x)]-[Af(x)+Bg(x)]}{\delta x}$$

$$= A\left[\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}\right] + B\left[\lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}\right].$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[Af(x) + Bg(x)] = A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

This is easily extended to **"linear combinations"** of three or more functions of x.

EXAMPLES

1. Write down the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = 6x^2 + 2x^6 + 13x - 7.$$

Solution

Using the linearity property, the standard derivative of x^n, and the derivative of a constant, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\frac{\mathrm{d}}{\mathrm{d}x}[x^2] + 2\frac{\mathrm{d}}{\mathrm{d}x}[x^6] + 13\frac{\mathrm{d}}{\mathrm{d}x}[x^1] - \frac{\mathrm{d}}{\mathrm{d}x}[7]$$

$$= 12x + 12x^5 + 13.$$

2. Write down the derivative with respect to x of the function

$$\frac{5}{x^2} - 4\sin x + 2\ln x.$$

Solution

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{5}{x^2} - 4\sin x + 2\ln x\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[5x^{-2} - 4\sin x + 2\ln x\right] \\
&= -10x^{-3} - 4\cos x + \frac{2}{x} \\
&= \frac{-10}{x^3} - 4\cos x + \frac{2}{x}.
\end{aligned}$$

(b) Composite Functions (or Functions of a Function)

(i) Functions of a Linear Function

Expressions like $(5x + 2)^{16}$, $\sin(2x + 3)$, $\ln(7 - 4x)$ may be called **"functions of a linear function"**.

The general form is

$$f(ax + b),$$

where a and b are constants.

In the above illustrations, $f(x)$ would be x^{16}, $\sin x$ and $\ln x$ respectively.

Suppose we write

$$y = f(u) \quad \text{where} \quad u = ax + b.$$

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δy in y and δu in u.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u}\frac{\delta u}{\delta x}.$$

Assuming that δy and δu tend to zero as δx tends to zero,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}.$$

This rule is called the **"Function of a Function Rule"**, **"Composite Function Rule"** or **"Chain Rule"**.

EXAMPLES

1. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = (5x + 2)^{16}$.
   Solution
   First write $y = u^{16}$ where $u = 5x + 2$.
   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = 16u^{15}$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 5$.
   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = 16u^{15}.5 = 80(5x + 2)^{15}$.
2. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sin(2x + 3)$.
   Solution
   First write $y = \sin u$ where $u = 2x + 3$.
   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$.
   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos u.2 = 2\cos(2x + 3)$.
3. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \ln(7 - 4x)$.
   Solution
   First write $y = \ln u$ where $u = 7 - 4x$.
   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = -4$.
   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{u}.(-4) = \frac{-4}{7-4x}$.

Note:
For quickness, treat $ax + b$ as if it were a single x, then multiply the final result by the constant value, a.

(ii) Functions of a Function in general

The formula

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}$$

may be used for the composite function

$$f[g(x)].$$

We write

$$y = f(u) \quad \text{where} \quad u = g(x),$$

then apply the formula.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = (x^2 + 7x - 3)^4.$$

Solution

Let $y = u^4$ where $u = x^2 + 7x - 3$.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 4u^3.(2x+7)$$

$$= 4(x^2 + 7x - 3)^3(2x + 7).$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \ln(x^2 - 3x + 1).$$

Solution

Let $y = \ln u$ where $u = x^2 - 3x + 1$.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u}.(2x - 3) = \frac{2x - 3}{x^2 - 3x + 1}.$$

3. Determine the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$ in the case when

$$y = 2\sin(5x^2 - 1) + 19x.$$

Solution

Suppose $z = 2\sin(5x^2 - 1)$.

Let $z = 2\sin u$, where $u = 5x^2 - 1$.

Then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos u.10x = 20x\cos(5x^2 - 1).$$

Hence, the complete derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 20x\cos(5x^2 - 1) + 19.$$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 20\cos 4 + 19 \simeq 5.927$

Calculator must be in **radian mode**.

Note:
For quickness, treat $g(x)$ as if it were a single x, then multiply by $g'(x)$

EXAMPLE

Determine the derivative of $\sin^3 x$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^3 x\right] =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(\sin x)^3\right] =$$

$$3(\sin x)^2 . \cos x =$$

$$3\sin^2 x . \cos x.$$