"JUST THE MATHS"

SLIDES NUMBER

10.2

DIFFERENTIATION 2
(Rates of change)

by

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UNIT 10.2 - DIFFERENTIATION 2

RATES OF CHANGE

10.2.1 INTRODUCTION

For the functional relationship

$$y = f(x),$$

we may plot y against x to obtain a curve (or straight line).

If y is the distance travelled, at time x, of a moving object, the rate of increase of y with respect to x becomes **speed**.

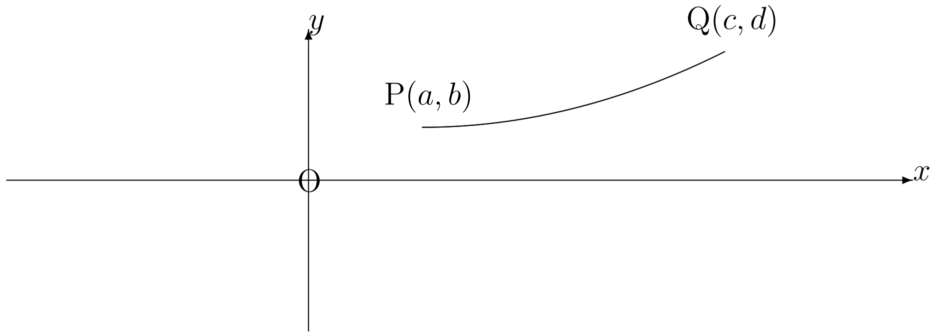
10.2.2 AVERAGE RATES OF CHANGE

For a vehicle travelling 280 miles in 7 hours,

$$\frac{280}{7} = 40$$

represents an **"average speed"** of 40 miles per hour over the whole journey.

Consider the relationship $y = f(x)$ between any two variables x and y.



Between P(a, b) and Q(c, d), an increase of $c - a$ in x gives rise to an increase of $d - b$ in y.

The average rate of increase of y with respect to x from P to Q is

$$\frac{d-b}{c-a}.$$

If y **decreases** as x increases (between P and Q), the average rate of increase will be negative

All rates of increase which are POSITIVE correspond to an INCREASING function.

All rates of increase which are NEGATIVE correspond to a DECREASING function.

Note:
For later work, $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ will denote points very close together.

The symbols δx and δy represent **"a small fraction of x"** and **"a small fraction of y"**, respectively.

δx is normally positive, but δy may turn out to be negative.

The average rate of increase may now be given by

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Average rate of increase =

$$\frac{(\text{new value of } y) \quad \text{minus} \quad (\text{old value of } y)}{(\text{new value of } x) \quad \text{minus} \quad (\text{old value of } x)}$$

EXAMPLE
Determine the average rate of increase of the function

$$y = x^2$$

between the following pairs of points on its graph:

(a) $(3, 9)$ and $(3.3, 10.89)$;

(b) $(3, 9)$ and $(3.2, 10.24)$;

(c) $(3, 9)$ and $(3.1, 9.61)$.

Solution

The results are

(a) $\frac{\delta y}{\delta x} = \frac{1.89}{0.3} = 6.3$;

(b) $\frac{\delta y}{\delta x} = \frac{1.24}{0.2} = 6.2$;

(c) $\frac{\delta y}{\delta x} = \frac{0.61}{0.1} = 6.1$

10.2.3 INSTANTANEOUS RATES OF CHANGE

Allowing Q to approach P along the curve, we may determine the **actual** rate of increase of y with respect to x at P.

The above solution suggests that the rate of increase of $y = x^2$ with respect to x at the point $(3, 9)$ is equal to 6.

This is called the **"instantaneous rate of increase of y with respect to x"** at the chosen point.

In general, we consider a limiting process in which an **infinite** number of points approach the chosen one along the curve.

The limiting process is represented by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

10.2.4 DERIVATIVES

(a) The Definition of a Derivative

If

$$y = f(x),$$

the **"derivative of y with respect to x"** at any point (x, y) on the graph of the function is defined to be the instantaneous rate of increase of y with respect to x at that point.

If a small increase of δx in x gives rise to a corresponding increase (positive or negative) of δy in y, the derivative will be given by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

This limiting value is usually denoted by one of the three symbols

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \quad f'(x) \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}[f(x)].$$

Notes:

1. $\frac{\mathrm{d}}{\mathrm{d}x}$ is called a **"differential operator"**;
2. $f'(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]$ are normally used when the second variable, y, is not involved.
3. The derivative of a constant function must be zero.
4. The derivative represents the **gradient of the tangent at the point** (x, y) to the curve whose equation is
$$y = f(x).$$

(b) Differentiation from First Principles

EXAMPLES

1. Differentiate the function x^4 from first principles.

   Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^4\right] = \lim_{\delta x \to 0} \frac{(x+\delta x)^4 - x^4}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^4 + 4x^3\delta x + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4 - x^4}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[4x^3 + 6x^2\delta x + 4x(\delta x)^2 + (\delta x)^3\right] = 4x^3.$$

Note:
The general formula is

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$$

for any constant value n, not necessarily an integer.

2. Differentiate the function $\sin x$ from first principles.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}.$$

But,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x.$$

3. Differentiate from first principles the function

$$\log_b x,$$

where b is any base of logarithms.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_b x\right] = \lim_{\delta x \to 0} \frac{\log_b(x+\delta x) - \log_b x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\log_b\left(1+\frac{\delta x}{x}\right)}{\delta x}.$$

But writing

$$\frac{\delta x}{x} = r \quad \text{that is} \quad \delta x = rx,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_b x\right] = \frac{1}{x}\lim_{r \to 0} \frac{\log_b(1+r)}{r}$$

$$= \frac{1}{x}\lim_{r \to 0} \log_b(1+r)^{\frac{1}{r}}.$$

For convenience, we may choose b so that the above limiting value is equal to 1.

This will occur when

$$b = \lim_{r \to 0}(1+r)^{\frac{1}{r}}.$$

The appropriate value of b turns out to be approximately 2.71828

This is the standard base of natural logarithms denoted by e.

Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_e x\right] = \frac{1}{x}.$$

Note:

In scientific work, the natural logarithm of x is usually denoted by $\ln x$.