“JUST THE MATHS”

SLIDES NUMBER

10.1

DIFFERENTIATION 1
(Functions and limits)

by

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10.1.1 Functional notation
10.1.2 Numerical evaluation of functions
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UNIT 10.1 - DIFFERENTIATION 1
FUNCTIONS AND LIMITS

10.1.1 FUNCTIONAL NOTATION

Introduction

If a variable quantity, y, depends for its values on another variable quantity, x, we say that **"y is a function of x"**.

In general, we write $y = f(x)$.

This is pronounced "y equals f of x".

Notes:
(i) y is called the **"dependent variable"** and x is called the **"independent variable"**.

(ii) We do not always use the letter f.

ILLUSTRATIONS

1. $P = P(T)$ could mean that a pressure, P, is a function of absolute temperature, T;

2. $i = i(t)$ could mean that an electric current, i, is a function of time t;

3. the original statement, $y = f(x)$ could have been written $y = y(x)$.

The general format:

DEPENDENT VARIABLE = DEPENDENT VARIABLE(INDEPENDENT VARIABLE)

10.1.2 NUMERICAL EVALUATION OF FUNCTIONS

If α is a number, then $f(\alpha)$ denotes the value of the function $f(x)$ when $x = \alpha$ is substituted into it.

ILLUSTRATION

If

$$f(x) \equiv 4\sin 3x,$$

then,

$$f\left(\frac{\pi}{4}\right) = 4\sin\frac{3\pi}{4} = 4 \times \frac{1}{\sqrt{2}} \cong 2.828$$

10.1.3 FUNCTIONS OF A LINEAR FUNCTION

The notation

$$f(ax+b),$$

where a and b are constants, implies a **known** function, $f(x)$, in which x has been replaced by the linear function $ax+b$.

ILLUSTRATION

If

$$f(x) \equiv 3x^2 - 7x + 4,$$

then,

$$f(5x-1) \equiv 3(5x-1)^2 - 7(5x-1) + 4.$$

It usually best to leave the expression in the bracketed form.

10.1.4 COMPOSITE FUNCTIONS (or Functions of a Function) IN GENERAL

The symbol

$$f[g(x)]$$

implies a **known** function, $f(x)$, in which x has been replaced by **another known** function, $g(x)$.

ILLUSTRATION

If

$$f(x) \equiv x^2 + 2x - 5$$

and

$$g(x) \equiv \sin x,$$

then,

$$f[g(x)] \equiv \sin^2 x + 2\sin x - 5;$$

but also,

$$g[f(x)] \equiv \sin(x^2 + 2x - 5),$$

which is not identical to the first result.

Hence, in general,

$$f[g(x)] \not\equiv g[f(x)].$$

Exceptions

If

$$f(x) \equiv e^x \quad \text{and} \quad g(x) \equiv \log_e x$$

we obtain

$$f[g(x)] \equiv e^{\log_e x} \equiv x \quad \text{and} \quad g[f(x)] \equiv \log_e(e^x) \equiv x.$$

The functions $\log_e x$ and e^x are said to be **"inverses"** of each other.

10.1.5 INDETERMINATE FORMS

Certain fractional expressions involving functions can reduce to

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

These forms are meaningless or **"indeterminate"**.

Indeterminate forms need to be dealt with using **"limiting values"**.

(a) The Indeterminate Form $\frac{0}{0}$

In the fractional expression

$$\frac{f(x)}{g(x)},$$

suppose that $f(\alpha) = 0$ and $g(\alpha) = 0$.

It is impossible to evaluate the fraction when $x = \alpha$.

We may consider its values as x becomes increasingly close to α with out actually reaching it

We say that **"x tends to α"**.

Note:

By the **Factor Theorem** (Unit 1.8), $(x - \alpha)$ must be a factor of both numerator and denominator.

The result as x approaches α is denoted by

$$\lim_{x \to \alpha} \frac{f(x)}{g(x)}.$$

EXAMPLE

Calculate

$$\lim_{x \to 1} \frac{x-1}{x^2+2x-3}.$$

Solution

First we factorise the denominator.

One of its factors must be $x-1$ because it takes the value zero when $x=1$.

The result is therefore

$$\lim_{x \to 1} \frac{x-1}{(x-1)(x+3)}$$

$$= \lim_{x \to 1} \frac{1}{x+3} = \frac{1}{4}.$$

(b) The Indeterminate Form $\frac{\infty}{\infty}$

Problem

To evaluate either

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

or

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)}.$$

EXAMPLE

Calculate

$$\lim_{x\to\infty} \frac{2x^2+3x-1}{7x^2-2x+5}.$$

Solution

We divide numerator and denominator by the highest power of x appearing.

$$\lim_{x\to\infty} \frac{2+\frac{3}{x}-\frac{1}{x^2}}{7-\frac{2}{x}+\frac{5}{x^2}} = \frac{2}{7}.$$

Notes:

(i) For the ratio of two polynomials of equal degree, the limiting value as $x \to \pm\infty$ is the ratio of the leading coefficients of x.

(ii) For two polynomials of unequal degree, we insert zero coefficients in appropriate places to consider them as being of equal degree.

The results then obtained will be either zero or infinity.

ILLUSTRATION

$$\lim_{x\to\infty} \frac{5x+11}{3x^2-4x+1} = \lim_{x\to\infty} \frac{0x^2+5x+11}{3x^2-4x+1} = \frac{0}{3} = 0.$$

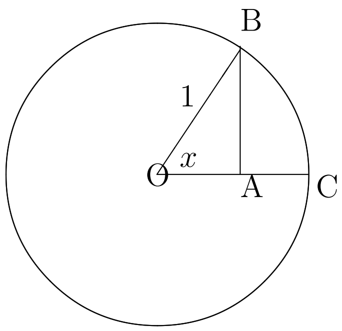
A Useful Standard Limit

In Unit 3.3, it is shown that, for very small values of x in radians, $\sin x \simeq x$.

This suggests that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

For a non-rigorous proof, consider the following diagram in which the angle x is situated at the centre of a circle with radius 1.



Length of line AB $= \sin x$.

Length of arc BC $= x$.

As x decreases almost to zero, these lengths become closer.

That is,

$$\sin x \to x \quad \text{as} \quad x \to 0$$

or

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

10.1.6 EVEN AND ODD FUNCTIONS

Introduction

Any **even** power of x will be unchanged in value if x is replaced by $-x$.

Any **odd** power of x will be unchanged in numerical value, though altered in sign, if x is replaced by $-x$.

DEFINITION

A function $f(x)$ is said to be **"even"** if it satisfies the identity

$$f(-x) \equiv f(x).$$

ILLUSTRATIONS: x^2, $2x^6 - 4x^2 + 5$, $\cos x$.

DEFINITION

A function $f(x)$ is aid to be **"odd"** if it satisfies the identity

$$f(-x) \equiv -f(x).$$

ILLUSTRATIONS: x^3, $x^5 - 3x^3 + 2x$, $\sin x$.

Note:

It is not necessary for every function to be either even or odd. For example, the function $x + 3$ is neither even nor odd.

EXAMPLE

Express an arbitrary function, $f(x)$ as the sum of an even function and an odd function.

Solution

We may write

$$f(x) \equiv \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

The first term on the R.H.S. is unchanged if x is replaced by $-x$.

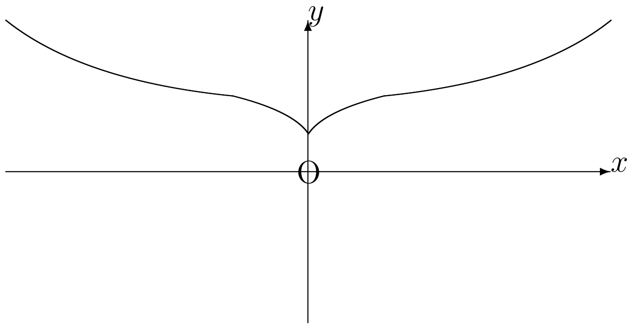
The second term on the R.H.S. is reversed in sign if x is replaced by $-x$.

We have thus expressed $f(x)$ as the sum of an even function and an odd function.

GRAPHS OF EVEN AND ODD FUNCTIONS

(i) The graph of the relationship $y = f(x)$, where $f(x)$ is **even**, will be symmetrical about the y-axis.

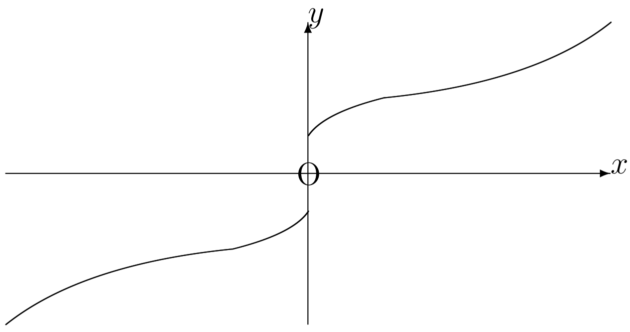
For every point (x, y) on the graph, there is also the point $(-x, y)$.



(ii) The graph of the relationship $y = f(x)$, where $f(x)$ is **odd**, will be symmetrical with respect to the origin.

For every point (x, y) on the graph, there is also the point $(-x, -y)$.

The part of the graph for $x < 0$ can be obtained from the part for $x > 0$ by reflecting it first in the x-axis and then in the y-axis.

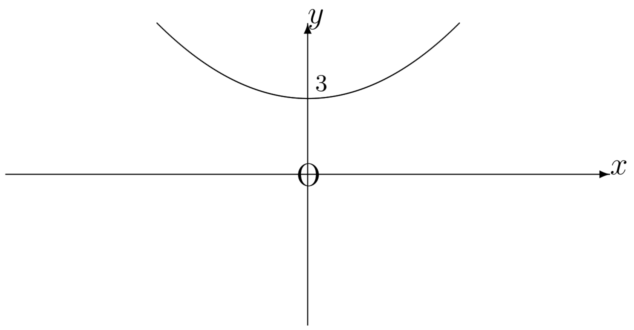


EXAMPLE

Sketch the graph, from $x = -3$ to $x = 3$ of the even function, $f(x)$, defined in the interval $0 < x < 3$ by the formula

$$f(x) \equiv 3 + x^3.$$

Solution



ALGEBRAICAL PROPERTIES OF ODD AND EVEN FUNCTIONS

1. The product of an even function and an odd function is an odd function.

Proof:

If $f(x)$ is even and $g(x)$ is odd, then

$$f(-x).g(-x) \equiv f(x).[-g(x)] \equiv -f(x).g(x).$$

2. The product of an even function and an even function is an even function.

Proof:

If $f(x)$ and $g(x)$ are both even functions, then

$$f(-x).g(-x) \equiv f(x).g(x).$$

3. The product of an odd function and an odd function is and even function.

Proof:

If $f(x)$ and $g(x)$ are both odd functions, then

$$f(-x).g(-x) \equiv [-f(x)].[-g(x)] \equiv f(x).g(x).$$

EXAMPLE

Show that the function

$$f(x) \equiv \sin^4 x. \tan x$$

is an **odd** function.

Solution

$$f(-x) \equiv \sin^4(-x). \tan(-x) \equiv -\sin^4 x. \tan x.$$