"JUST THE MATHS"

SLIDES NUMBER

1.8

ALGEBRA 8
(Polynomials)

by

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UNIT 1.8 - ALGEBRA 8
POLYNOMIALS

Introduction

General form,

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n,$$

a **"polynomial of degree n in x"**, having **"coefficients"** $a_0, a_1, a_2, a_3, ...a_n$, usually constant.

Note:
Polynomials of degree 1, 2 and 3 are called respectively "linear", "quadratic" and "cubic" polynomials.

1.8.1 THE FACTOR THEOREM

If $P(x)$ denotes an algebraic polynomial which has the value zero when $x = \alpha$, then $x - \alpha$ is a factor of the polynomial and
$P(x) \equiv (x - \alpha)\times$ another polynomial, $Q(x)$, of one degree lower.

$x = \alpha$ is called a **"root"** of the polynomial.

1.8.2 APPLICATION TO QUADRATIC AND CUBIC EXPRESSIONS

(a) Quadratic Expressions

To locate a root, try $x = 0, 1, -1, 2, -2, 3, -3, 4, -4, ...$

EXAMPLES

1. $x^2 + 2x - 3$ is zero when $x = 1$; hence $x - 1$ is a factor.
   The complete factorisation is $(x - 1)(x + 3)$.
2. $3x^2 + 20x - 7$ is zero when $x = -7$; hence $(x + 7)$ is a factor.
   The complete factorisation is
   $$(x + 7)(3x - 1).$$

(b) Cubic Expressions

Standard form is

$$ax^3 + bx^2 + cx + d.$$

EXAMPLES

1. $x^3 + 3x^2 - x - 3$ is zero when $x = 1$.
   Hence, $(x - 1)$ is a factor.

Thus,

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(px^2 + qx + r)$$

for some constants p, q and r.
Comparing coefficients on both sides,

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x^2 + 4x + 3)$$

$$\equiv (x - 1)(x + 1)(x + 3).$$

2. $x^3 + 4x^2 + 4x + 1$ is zero when $x = -1$ and so $x + 1$ must be a factor.

Hence

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)(px^2 + qx + r)$$

for some constants p, q and r.
Comparing coefficients on both sides,

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)(x^2 + 3x + 1).$$

1.8.3 CUBIC EQUATIONS

EXAMPLES

1. Solve the cubic equation

$$x^3 + 3x^2 - x - 3 = 0.$$

Solution

One solution is $x = 1$ and so $(x - 1)$ must be a factor.

$$(x - 1)(x^2 + 4x + 3) = 0;$$

$$(x - 1)(x + 1)(x + 3) = 0.$$

Solutions are $x = 1$, $x = -1$ and $x = -3$.

2. Solve the cubic equation

$$2x^3 - 7x^2 + 5x + 54 = 0.$$

Solution

One solution is $x = -2$ and so $(x+2)$ must be a factor.

$$(x + 2)(2x^2 - 11x + 27) = 0;$$

$$x = -2 \ \text{ or } \ x = \frac{11 \pm \sqrt{121 - 216}}{4} \ \text{ not real.}$$

1.8.4 LONG DIVISION OF POLYNOMIALS

(a) Exact Division

EXAMPLES

1. Divide the cubic expression $x^3 + 3x^2 - x - 3$ by $x - 1$.

Solution

$$\begin{array}{r|l} & x^2 + 4x + 3 \\ \hline x - 1 & x^3 + 3x^2 - x - 3 \\ & \underline{x^3 - x^2} \\ & \quad 4x^2 - x - 3 \\ & \quad \underline{4x^2 - 4x} \\ & \qquad\quad 3x - 3 \\ & \qquad\quad \underline{3x - 3} \\ & \qquad\qquad\quad 0 \end{array}$$

Hence,

$$x^3 + 3x^2 - x - 3 \equiv (x-1)(x^2+4x+3) \equiv (x-1)(x+1)(x+3)$$

2. Solve, completely, the cubic equation

$$x^3 + 4x^2 + 4x + 1 = 0$$

Solution

One solution is $x = -1$ so that $(x + 1)$ is a factor.

$$
\begin{array}{r}
x^2 + 3x + 1 \\
x+1 \overline{) x^3 + 4x^2 + 4x + 1} \\
\underline{x^3 + x^2 \qquad\qquad\quad} \\
3x^2 + 4x + 1 \\
\underline{3x^2 + 3x \qquad} \\
x + 1 \\
\underline{x + 1} \\
0
\end{array}
$$

Hence,

$$(x+1)(x^2+3x+1) = 0;$$

$x = -1$ and $x = \frac{-3 \pm \sqrt{9-4}}{2} \simeq -0.382 \quad \text{or} \quad -2.618$

(b) Non-exact Division

Here, the remainder will not be zero.

EXAMPLES

1. Divide the polynomial $6x + 5$ by the polynomial $3x - 1$

   Solution

$$
\begin{array}{r}
2 \\
3x-1 \overline{) 6x + 5} \\
\underline{6x - 2} \\
7
\end{array}
$$

Hence,

$$\frac{6x+5}{3x-1} \equiv 2 + \frac{7}{3x-1}.$$

2. Divide $3x^2 + 2x$ by $x + 1$.

Solution

$$\begin{array}{r@{}l}
 & 3x - 1 \\
x+1 \big) & \overline{3x^2 + 2x} \\
 & \underline{3x^2 + 3x} \\
 & \qquad -x \\
 & \qquad \underline{-x - 1} \\
 & \qquad\quad 1
\end{array}$$

Hence,

$$\frac{3x^2+2x}{x+1} \equiv 3x - 1 + \frac{1}{x+1}.$$

3. Divide $x^4 + 2x^3 - 2x^2 + 4x - 1$ by $x^2 + 2x - 3$.

Solution

$$\begin{array}{r@{}l}
 & x^2 \qquad\quad + 1 \\
x^2+2x-3 \big) & \overline{x^4 + 2x^3 - 2x^2 + 4x - 1} \\
 & \underline{x^4 + 2x^3 - 3x^2} \\
 & \qquad\qquad x^2 + 4x - 1 \\
 & \qquad\qquad \underline{x^2 + 2x - 3} \\
 & \qquad\qquad\qquad 2x + 2
\end{array}$$

Hence

$$\frac{x^4+2x^3-2x^2+4x-1}{x^2+2x-3} \equiv x^2 + 1 + \frac{2x+2}{x^2+2x-3}.$$