"JUST THE MATHS"

SLIDES NUMBER

1.5

ALGEBRA 5
(Manipulation of algebraic expressions)

by

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UNIT 1.5 - ALGEBRA 5
MANIPULATION OF ALGEBRAIC EXPRESSIONS

1.5.1 SIMPLIFICATION OF EXPRESSIONS

Remove brackets and collect together any terms which have the same format

Elementary Illustrations

1. $a + a + a + 3 + b + b + b + b + 8 \equiv 3a + 4b + 11.$
2. $11p^2 + 5q^7 - 8p^2 + q^7 \equiv 3p^2 + 6q^7.$
3. $a.(2a - b) + b.(a + 5b) - a^2 - 4b^2 \equiv 2a^2 - a.b + b.a + 5b^2 - a^2 - 4b^2 \equiv a^2 + b^2.$

Further illustrations

1. $x(2x + 5) + x^2(3 - x) \equiv 2x^2 + 5x + 3x^2 - x^3 \equiv 5x^2 + 5x - x^3.$
2. $x^{-1}(4x - x^2) - 6(1 - 3x) \equiv 4 - x - 6 + 18x \equiv 17x - 2.$

Two or more brackets multiplied together

$(a+b)(c+d) = (a+b)c + (a+b)d = ac + bc + ad + bd.$

EXAMPLES:

1. $(x+3)(x-5) \equiv x^2 + 3x - 5x - 15 \equiv x^2 - 2x - 15.$
2. $(x^3 - x)(x+5) \equiv x^4 - x^2 + 5x^3 - 5x.$
3. $(x+a)^2 \equiv (x+a)(x+a) \equiv x^2 + ax + ax + a^2 \equiv x^2 + 2ax + a^2$; a **"Perfect Square"**.
4. $(x+a)(x-a) \equiv x^2 + ax - ax - a^2 \equiv x^2 - a^2$; the **"Difference of two squares"**.

1.5.2 FACTORISATION

Introduction

"Factor" means "Multiplier".

Examples

1. $3x + 12 \equiv 3(x+4).$
2. $8x^2 - 12x \equiv x(8x - 12) \equiv 4x(2x - 3).$
3. $5x^2 + 15x^3 \equiv x^2(5 + 15x) \equiv 5x^2(1 + 3x).$
4. $6x + 3x^2 + 9xy \equiv x(6 + 3x + 9y) \equiv 3x(2 + x + 3y).$

Note:
When none of the factors can be broken down into simpler factors, the original expression is said to have been factorised into **"irreducible factors"**.

Factorisation of quadratic expressions

A "Quadratic Expression" is an expression of the form

$$ax^2 + bx + c.$$

The quadratic expression has "coefficients a, b and c".

EXAMPLES:

(a) When the coefficient of x^2 is 1

1. $x^2 + 5x + 6 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn$;
   $5 = m + n$ and $6 = mn$;
   By inspection, $m = 2$ and $n = 3$.
   Hence $x^2 + 5x + 6 \equiv (x + 2)(x + 3)$.
2. $x^2 + 4x - 21 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn$;
   $4 = m + n$ and $-21 = mn$;
   By inspection, $m = -3$ and $n = 7$.
   Hence $x^2 + 4x - 21 \equiv (x - 3)(x + 7)$.

Note:
For simple cases, carry out the factorisation entirely by inspection.

$$x^2 + 2x - 8 \equiv (x+?)(x+?) \equiv (x-2)(x+4).$$

$$x^2 + 10x + 25 \equiv (x+5)^2.$$

$$x^2 - 64 \equiv (x-8)(x+8).$$

$$x^2 - 13x + 2 \quad \text{won't} \quad \text{factorise.}$$

(b) When the coefficient of x^2 is not 1

Determine the possible pairs of factors of the coefficient of x^2 and the possible pairs of factors of the constant term.

EXAMPLES

1. To factorise the expression $2x^2 + 11x + 12$,
   Try $(2x+1)(x+12)$, $(2x+12)(x+1)$, $(2x+6)(x+2)$, $(2x+2)(x+6)$, $(2x+4)(x+3)$ and $(2x+3)(x+4)$.

$$2x^2 + 11x + 12 \equiv (2x+3)(x+4).$$

2. To factorise the expression $6x^2 + 7x - 3$,

Try $(6x+3)(x-1)$, $(6x-3)(x+1)$, $(6x+1)(x-3)$, $(6x-1)(x+3)$, $(3x+3)(2x-1)$, $(3x-3)(2x+1)$, $(3x+1)(2x-3)$ and $(3x-1)(2x+3)$.

$$6x^2 + 7x - 3 \equiv (3x-1)(2x+3).$$

1.5.3 COMPLETING THE SQUARE IN A QUADRATIC EXPRESSION

We use

$$(x+a)^2 \equiv x^2 + 2ax + a^2$$

and

$$(x-a)^2 \equiv x^2 - 2ax + a^2.$$

ILLUSTRATIONS

1. $x^2 + 6x + 9 \equiv (x+3)^2$.
2. $x^2 - 8x + 16 \equiv (x-4)^2$.
3. $4x^2 - 4x + 1 \equiv 4\left[x^2 - x + \frac{1}{4}\right] \equiv 4\left(x - \frac{1}{2}\right)^2$.
4. $x^2 + 6x + 11 \equiv (x+3)^2 + 2$.
5. $x^2 - 8x + 7 \equiv (x-4)^2 - 9$.
6. $4x^2 - 4x + 5 \equiv 4\left[x^2 - x + \frac{5}{4}\right]$

$$\equiv 4\left[\left(x-\tfrac{1}{2}\right)^2-\tfrac{1}{4}+\tfrac{5}{4}\right]$$

$$\equiv 4\left[\left(x-\tfrac{1}{2}\right)^2+1\right]$$

$$\equiv 4\left(x-\frac{1}{2}\right)^2+4.$$

1.5.4 ALGEBRAIC FRACTIONS

Revision

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd},\qquad \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd},\qquad \frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$$

EXAMPLES

1.

$$\frac{5}{25+15x}\equiv\frac{1}{5+3x},$$

assuming that $x\neq-\frac{5}{3}$.

2.

$$\frac{4x}{3x^2+x}\equiv\frac{4}{3x+1},$$

assuming that $x\neq 0$ or $-\frac{1}{3}$.

3.

$$\frac{x+2}{x^2+3x+2}\equiv\frac{x+2}{(x+2)(x+1)}\equiv\frac{1}{x+1},$$

assuming that $x\neq-1$ or -2.

4.

$$\frac{3x+6}{x^2+3x+2} \times \frac{x+1}{2x+8}$$

$$\equiv \frac{3(x+2)(x+1)}{2(x+4)(x+1)(x+2)}$$

$$\equiv \frac{3}{2(x+4)},$$

assuming that $x \neq -1, -2$ or -4.

5.

$$\frac{3}{x+2} \div \frac{x}{2x+4}$$

$$\equiv \frac{3}{x+2} \times \frac{2x+4}{x}$$

$$\equiv \frac{3}{x+2} \times \frac{2(x+2)}{x} \equiv \frac{6}{x},$$

assuming that $x \neq 0$ or -2.

6.

$$\frac{4}{x+y} - \frac{3}{y}$$

$$\equiv \frac{4y - 3(x+y)}{(x+y)y}$$

$$\equiv \frac{y-3x}{(x+y)y}.$$

7.

$$\frac{x}{x+1} + \frac{4-x^2}{x^2-x-2}$$

$$\equiv \frac{x(x-2)}{(x+1)(x-2)} + \frac{4-x^2}{(x+1)(x-2)}$$

$$\equiv \frac{x^2-2x+4-x^2}{(x+1)(x-2)}$$

$$\equiv \frac{2(2-x)}{(x+1)(x-2)} = -\frac{2}{x+1}$$

assuming that $x \neq 2 \text{ or } -1.$