“JUST THE MATHS”

SLIDES NUMBER

1.4

ALGEBRA 4
(Logarithms)

by

A.J.Hobson

UNIT 1.4 - ALGEBRA 4 - LOGARITHMS

1.4.1 COMMON LOGARITHMS

We normally count with a base of 10.

Each successive digit of a number corresponds to that digit multiplied by a certain power of 10.

ILLUSTRATION

$$73,520 = 7 \times 10^4 + 3 \times 10^3 + 5 \times 10^2 + 2 \times 10^1.$$

Problem

Can a given number can be expressed as a single power of 10, not necessarily an integer power ?

The power will need to be **positive** since powers of 10 are not normally negative (or even zero).

ILLUSTRATION

By calculator,

$$1.99526 \simeq 10^{0.3}$$

and

$$2 \simeq 10^{0.30103}.$$

DEFINITION

In general, when

$$x = 10^y,$$

for some positive number x, we say that y is the **"logarithm to base 10"** of x (or " **Common Logarithm"** of x) and we write

$$y = \log_{10} x.$$

ILLUSTRATIONS

1. $\log_{10} 1.99526 = 0.3$ from the earlier illustrations.
2. $\log_{10} 2 = 0.30103$ from the earlier illustrations.
3. $\log_{10} 1 = 0$ simply because $10^0 = 1$.

1.4.2 LOGARITHMS IN GENERAL

DEFINITION

If B is a fixed positive number and x is another positive number such that

$$x = B^y,$$

we say that y is the "logarithm to base B of x and we write

$$y = \log_B x.$$

ILLUSTRATIONS

1. $\log_B 1 = 0$ simply because $B^0 = 1$.
2. $\log_B B = 1$ simply because $B^1 = B$.
3. $\log_B 0$ doesn't really exist because no power of B could ever be equal to zero.

   But, since a very large negative power of B will be a very small positive number, we usually write

$$\log_B 0 = -\infty$$

1.4.3 USEFUL RESULTS

(a) For any positive number x,

$$x = B^{\log_B x}.$$

Proof

In $x = B^y$, replace y by $\log_B x$.

(b) For any number y,

$$y = \log_B B^y.$$

In $y = \log_B x$, replace x by B^y.

1.4.4 PROPERTIES OF LOGARITHMS

(a) The Logarithm of Product.

$$\log_B p.q = \log_B p + \log_B q.$$

Proof:

From Result (a) of the previous section,

$$p.q = B^{\log_B p}.B^{\log_B q} = B^{\log_B p + \log_B q}.$$

(b) The Logarithm of a Quotient

$$\log_B \frac{p}{q} = \log_B p - \log_B q.$$

Proof:

From Result (a) of the previous section,

$$\frac{p}{q} = \frac{B^{\log_B p}}{B^{\log_B q}} = B^{\log_B p - \log_B q}.$$

(c) The Logarithm of an Exponential

$$\log_B p^n = n \log_B p,$$

where n need not be an integer.

Proof:

From Result (a) of the previous section,

$$p^n = \left(B^{\log_B p}\right)^n = B^{n \log_B p}.$$

(d) The Logarithm of a Reciprocal

$$\log_B \frac{1}{q} = -\log_B q.$$

Proof:

Method 1.
Left-hand side =
$\log_B 1 - \log_B q = 0 - \log_B q = -\log_B q$

Method 2.
Left-hand side = $\log_B q^{-1} = -\log_B q$.

(e) Change of Base

$$\log_B x = \frac{\log_A x}{\log_A B}.$$

Proof:

Suppose $y = \log_B x$.

Then, $x = B^y$.

Hence,

$$\log_A x = \log_A B^y = y \log_A B.$$

Thus

$$y = \frac{\log_A x}{\log_A B}.$$

Note:
Logarithms to any base are directly proportional to logarithms to another base.

1.4.5 NATURAL LOGARITHMS

In scientific work, only two bases of logarithms are used.

One is base 10 (for **"Common"** Logarithms).

The other is a base $e = 2.71828......$ (for **"Natural"** Logarithms) arising out of calculus.

The Natural Logarithm of x is denoted by $\log_e x$ or $\ln x$.

Note:
By change of base formula,

$$\log_{10} x = \frac{\log_e x}{\log_e 10} \qquad \text{and} \qquad \log_e x = \frac{\log_{10} x}{\log_{10} e}.$$

EXAMPLES

1. Solve for x the “indicial equation”

$$4^{3x-2} = 26^{x+1}.$$

Solution

Take logarithms of both sides,

$$\begin{gathered}
(3x-2)\log_{10} 4 = (x+1)\log_{10} 26; \\
(3x-2)0.6021 = (x+1)1.4150; \\
1.8063x - 1.2042 = 1.4150x + 1.4150; \\
(1.8603 - 1.4150)x = 1.4150 + 1.2042; \\
0.3913x = 2.6192; \\
x = \frac{2.6192}{0.3913} \simeq 6.6936
\end{gathered}$$

2. Rewrite the expression

$$4x + \log_{10}(x+1) - \log_{10} x - \frac{1}{2}\log_{10}(x^3 + 2x^2 - x)$$

as the common logarithm of a single mathematical expression.

Solution

Convert every term to 1× a logarithm.

$$4x = \log_{10} 10^{4x}$$

$$\frac{1}{2}\log_{10}(x^3 + 2x^2 - x) = \log_{10}\left(x^3 + 2x^2 - x\right)^{\frac{1}{2}}.$$

Hence,

$$\log_{10} \frac{10^{4x}(x+1)}{x\sqrt{(x^3 + 2x^2 - x)}}.$$

3. Rewrite without logarithms the equation

$$2x + \ln x = \ln(x-7).$$

Solution

Convert both sides to the natural logarithm of a single mathematical expression.

$$\text{L.H.S.} = 2x + \ln x = \ln e^{2x} + \ln x = \ln xe^{2x}.$$

Hence,

$$xe^{2x} = x - 7.$$

4. Solve for x the equation

$$6 \ln 4 + \ln 2 = 3 + \ln x.$$

Solution

Using $6 \ln 4 = \ln 4^6$ and $3 = \ln e^3$,

$$\ln 2(4^6) = \ln xe^3.$$

Hence,

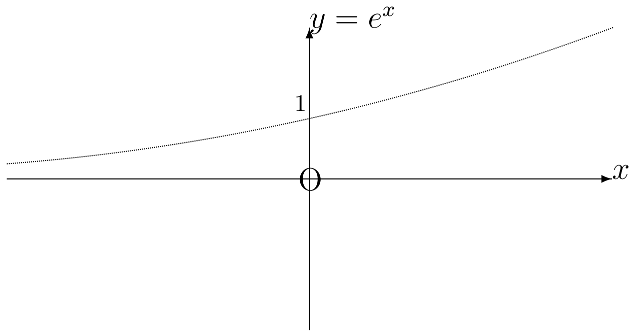
$$2(4^6) = xe^3,$$

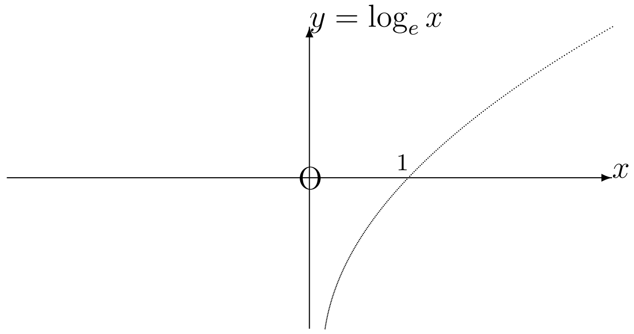
so that

$$x = \frac{2(4^6)}{e^3} \simeq 407.856$$

1.4.6 GRAPHS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

The graphs of $y = e^x$ and $y = \log_e x$ are as follows:





1.4.7 LOGARITHMIC SCALES

In a certain kind of graphical work, some use is made of a linear scale along which numbers can be allocated according to their logarithmic distances from a chosen origin of measurement.

For logarithms to base 10, the number 1 is placed at the zero of measurement (since $\log_{10} 1 = 0$).

The number 10 is placed at the first unit of measurement (since $\log_{10} 10 = 1$).

The number 100 is placed at the second unit of measurement (since $\log_{10} 100 = 2$) and so on.

Numbers such as $10^{-1} = 0.1$, $10^{-2} = 0.01$ etc. are placed at the points corresponding to -1 and -2 etc. respectively on an ordinary linear scale.

The logarithmic scale appears in **"cycles"**

Each cycle corresponds to a range of numbers between two consecutive powers of 10.

Intermediate numbers are placed at intervals which correspond to their logarithm values.

0.1 0.2 0.3 0.4 1 2 3 4 10

Notes:
(i) A given set of numbers will determine how many cycles are required.

For example .3, .6, 5, 9, 23, 42, 166 will require **four** cycles.

(ii) Commercially printed logarithmic scales do not specify the base of logarithms.