"JUST THE MATHS"

SLIDES NUMBER

1.3

ALGEBRA 3
(Indices and radicals (or surds))

by

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UNIT 1.3 - ALGEBRA 3
INDICES AND RADICALS (or Surds)

1.3.1 INDICES

(a) Positive Integer Indices

Let a and b be arbitrary numbers
Let m and n be natural numbers

Law No. 1

$$a^m \times a^n = a^{m+n}$$

Law No. 2

$$a^m \div a^n = a^{m-n}$$

assuming m greater than n.

Note:

$\frac{a^m}{a^m} = 1$ and $\frac{a^m}{a^m} = a^{m-m} = a^0$.

Hence, we **define** a^0 to be equal to 1.

Law No. 3

$$(a^m)^n = a^{mn}$$

$$a^m b^m = (ab)^m$$

EXAMPLE

Simplify the expression,

$$\frac{x^2y^3}{z} \div \frac{xy}{z^5}.$$

Solution

The expression becomes

$$\frac{x^2y^3}{z} \times \frac{z^5}{xy} = xy^2z^4.$$

(b) Negative Integer Indices

Law No. 4

$$a^{-1} = \frac{1}{a}$$

Note:

$\frac{a^m}{a^{m+1}} = \frac{1}{a}$ and $a^{m-[m+1]} = a^{-1}$.

Law No. 5

$$a^{-n} = \frac{1}{a^n}$$

Note:
$\frac{a^m}{a^{m+n}} = \frac{1}{a^n}$ and $a^{m-[m+n]} = a^{-n}$

Law No. 6

$$a^{-\infty} = 0$$

EXAMPLE

Simplify the expression,

$$\frac{x^5y^2z^{-3}}{x^{-1}y^4z^5} \div \frac{z^2x^2}{y^{-1}}.$$

Solution

The expression becomes

$$x^5y^2z^{-3}xy^{-4}z^{-5}y^{-1}z^{-2}x^{-2} = x^4y^{-3}z^{-10}.$$

(c) Rational Indices

(i) Indices of the form $\frac{1}{n}$ where n is a natural number.

$a^{\frac{1}{n}}$ means a number which gives a when it is raised to the power n. It is called an "n-th Root of a" and, sometimes there is more than one value.

ILLUSTRATION

$81^{\frac{1}{4}} = \pm 3$ but $(-27)^{\frac{1}{3}} = -3$ only

(ii) Indices of the form $\frac{m}{n}$ where m and n are natural numbers with no common factor.

$y^{\frac{m}{n}} = (y^m)^{\frac{1}{n}}$ or $(y^{\frac{1}{n}})^m$.

ILLUSTRATION

$27^{\frac{2}{3}} = 3^2 = 9$ or $27^{\frac{2}{3}} = 729^{\frac{1}{3}} = 9$

Note:
It may be shown that all of the standard laws of indices may be used for fractional indices.

1.3.2 RADICALS (or Surds)

"$\sqrt{\ }$" denotes the positive or **principal** square root of a number.

eg. $\sqrt{16} = 4$ and $\sqrt{25} = 5$.

The number under the radical is called the **RADICAND**

The **principal n-th root** of a number a is $\sqrt[n]{a}$

n is the **index** of the radical.

ILLUSTRATIONS

1. $\sqrt[3]{64} = 4$ since $4^3 = 64$
2. $\sqrt[3]{-64} = -4$ since $(-4)^3 = -64$

3. $\sqrt[4]{81} = 3$ since $3^4 = 81$
4. $\sqrt[5]{32} = 2$ since $2^5 = 32$
5. $\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$

Note:
If the index of the radical is an even number, then the radicand may not be negative.

(d) Rules for Square Roots

(i) $(\sqrt{a})^2 = a$

(ii) $\sqrt{a^2} = |a|$

(iii) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(iv) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

assuming that all the radicals can be evaluated

ILLUSTRATIONS

1. $\sqrt{9 \times 4} = \sqrt{36} = 6$ and $\sqrt{9} \times \sqrt{4} = 3 \times 2 = 6$
2. $\sqrt{\frac{144}{36}} = \sqrt{4} = 2$ and $\frac{\sqrt{144}}{\sqrt{36}} = \frac{12}{6} = 2.$

(e) Rationalisation of Radical (or Surd) Expressions.

EXAMPLES

1. Rationalise the surd form $\frac{5}{4\sqrt{3}}$

   Solution

$$\frac{5}{4\sqrt{3}} = \frac{5}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{12}$$

2. Rationalise the surd form $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$

   Solution

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}} \times \frac{\sqrt[3]{b^2}}{\sqrt[3]{b^2}} = \frac{\sqrt[3]{ab^2}}{\sqrt[3]{b^3}} = \frac{\sqrt[3]{ab^2}}{b}$$

3. Rationalise the surd form $\frac{4}{\sqrt{5}+\sqrt{2}}$

   Solution

   We use $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$

$$\frac{4}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{4\sqrt{5}-4\sqrt{2}}{3}$$

4. Rationalise the surd form $\frac{1}{\sqrt{3}-1}$

   Solution

$$\frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2}$$

(f) Changing numbers to and from radical form

$$| a^{\frac{m}{n}} | = \sqrt[n]{a^m}$$

EXAMPLES

1. Express the number $x^{\frac{2}{5}}$ in radical form

   Solution

   $$\text{Answer} = \sqrt[5]{x^2}$$

2. Express the number $\sqrt[3]{a^5b^4}$ in exponential form

   Solution

   $$\sqrt[3]{a^5b^4} = (a^5b^4)^{\frac{1}{3}} = a^{\frac{5}{3}}b^{\frac{4}{3}}$$