“JUST THE MATHS”

UNIT NUMBER

9.8

MATRICES 8
(Characteristic properties)
&
(Similarity transformations)

by

A.J.Hobson

9.8.1 Properties of eigenvalues and eigenvectors
9.8.2 Similar matrices
9.8.3 Exercises
9.8.4 Answers to exercises

UNIT 9.8 - MATRICES 8

CHARACTERISTIC PROPERTIES AND SIMILARITY TRANSFORMATIONS

9.8.1 PROPERTIES OF EIGENVALUES AND EIGENVECTORS

We list, here, a number of standard properties, together with either their proofs or an illustration of their proofs.

(i) The eigenvalues of a matrix are the same as those of its transpose.

Proof:

Given a square matrix, A, the eigenvalues of A^{T} are the solutions of the equation

$$|\mathrm{A}^{\mathrm{T}} - \lambda \mathrm{I}| = 0.$$

But, since I is a symmetric matrix, this is equivalent to

$$|(\mathrm{A} - \lambda \mathrm{I})^{\mathrm{T}}| = 0.$$

The result follows since a determinant is unchanged in value when it is transposed.

(ii) The eigenvalues of the multiplicative inverse of a matrix are the reciprocals of the eigenvalues of the matrix itself.

Proof:

If λ is any eigenvalue of a square matrix, A, then

$$\mathrm{AX} = \lambda \mathrm{X},$$

for some column vector, X.

Premultiplying this relationship by A^{-1}, we obtain

$$\mathrm{A}^{-1}\mathrm{AX} = \mathrm{A}^{-1}(\lambda \mathrm{X}) = \lambda(\mathrm{A}^{-1}\mathrm{X}).$$

Thus,

$$\mathrm{A}^{-1}\mathrm{X} = \frac{1}{\lambda}\mathrm{X}.$$

(iii) The eigenvectors of a matrix and its multiplicative inverse are the same.

Proof:

This follows from the proof of **(ii)**, since

$$\mathrm{A}^{-1}\mathrm{X} = \frac{1}{\lambda}\mathrm{X}$$

implies that X is an eigenvector of A^{-1}.

(iv) If a matrix is multiplied by a single number, the eigenvalues are multiplied by that number, but the eigenvectors remain the same.

Proof:

If A is multiplied by α, we may write the equation $\mathrm{AX} = \lambda\mathrm{X}$ in the form $\alpha\mathrm{AX} = \alpha\lambda\mathrm{X}$.

Thus, $\alpha\mathrm{A}$ has eigenvalues, $\alpha\lambda$, and eigenvectors, X.

(v) If $\lambda_1,\ \ \lambda_2,\ \ \lambda_3,\ . . $ are the eigenvalues of the matrix A and n is a positive integer, then $\lambda_1^n,\ \ \lambda_2^n,\ \ \lambda_3^n,\ . .$ are the eigenvalues of A^n.

Proof:

If λ denotes any one of the eigenvalues of the matrix, A, then $\mathrm{AX} = \lambda\mathrm{X}$.

Premultiplying both sides by A, we obtain $\mathrm{A}^2\mathrm{X} = \mathrm{A}\lambda\mathrm{X} = \lambda\mathrm{A}X = \lambda^2\mathrm{X}$

Hence, λ^2 is an eigenvlaue of A^2.

Similarly, $\mathrm{A}^3\mathrm{X} = \lambda^3\mathrm{X}$, and so on.

(vi) If $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ . . .$ are the eigenvalues of the $n \times n$ matrix A, I is the $n \times n$ multiplicative identity matrix and k is a single number, then the eigenvalues of the matrix $\mathrm{A} + k\mathrm{I}$ are $\lambda_1 + k,\ \ \lambda_2 + k,\ \ \lambda_3 + k,\ . . .$.

Proof:

If λ is any eigenvalue of A, then $\mathrm{AX} = \lambda \mathrm{X}$.

Hence,

$$(\mathrm{A} + k\mathrm{I})\mathrm{X} = \mathrm{AX} + k\mathrm{X} = \lambda\mathrm{X} + k\mathrm{X} = (\lambda + k)\mathrm{X}.$$

(vii) A matrix is singular ($|\mathrm{A}| = 0$) if and only if at least one eigenvalue is equal to zero.

Proof:

(a) If X is an eigenvector corresponding to an eigenvalue, $\lambda = 0$, then $\mathrm{AX} = \lambda\mathrm{X} = [0]$.

From the theory of homogeneous linear equations (see Unit 7.4), it follows that $|\mathrm{A}| = 0$.

(b) Conversely, if $|\mathrm{A}| = 0$, the homogeneous system $\mathrm{AX} = [0]$ has a solution for X other than $\mathrm{X} = [0]$. Hence, at least one eigenvalue must be zero.

(viii) If A is an orthogonal matrix ($\mathrm{AA}^{\mathrm{T}} = \mathrm{I}$), then every eigenvalue is either $+1$ or -1.

Proof:

The statement $\mathrm{AA}^{\mathrm{T}} = \mathrm{I}$ can be written $\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{T}}$ so that, by **(i)** and **(ii)**, the eigenvalues of A are equal to their own reciprocals.

That is, they must have values $+1$ or -1.

(ix) If the elements of a matrix below the leading diagonal or the elements above the leading diagonal are all equal zero, then the eigenvalues are equal to the diagonal elements.

ILLUSTRATION

An **"upper-triangular matrix"**, A, of order 3 × 3, has the form

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}.$$

The characteristic equation is given by

$$0 = |\mathrm{A} - \lambda\mathrm{I}| = \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ 0 & b_2 - \lambda & c_2 \\ 0 & 0 & c_3 - \lambda \end{vmatrix} = (a_1 - \lambda)(b_2 - \lambda)(c_3 - \lambda).$$

Hence, $\lambda = a_1,\ b_2$ or c_3.

A similar proof holds for a **"lower-triangular matrix"**.

Note:
A special case of both a lower-triangular matrix and an upper-triangular matrix is a diagonal matrix.

(x) The sum of the eigenvalues of a matrix is equal to the trace of the matrix (the sum of the diagonal elements) and the product of the eigenvalues is equal to the determinant of the matrix.

ILLUSTRATION

We consider the case of a 2 × 2 matrix, A, given by

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

The characteristic equation is

$$0 = \begin{vmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{vmatrix} = \lambda^2 - (a_1 + b_2)\lambda + (a_1 b_2 - a_2 b_1).$$

But, for any quadratic equation, $a\lambda^2 + b\lambda + c = 0$, the sum of the solutions is equal to $-b/a$ and the product of the solutions is equal to c/a.

In this case, therefore, the sum of the solutions is $a_1 + b_2$, while the product of the solutions is $a_1 b_2 - a_2 b_1$.

9.8.2 SIMILAR MATRICES

In the previous section, a matrix and its transpose illustrated how two matrices can have the same eigenvalues. In this section, we deal with a more general case of this occurrence.

DEFINITION

Two matrices, A and B, are said to be **"similar"** if

$$\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP},$$

for some non-singular matrix, P.

Notes:

(i) P is certainly square, so that A and B must also be square and of the same order as P.

(ii) The relationship $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}$ is regarded as a **"transformation"** of the matrix, A, into the matrix, B.

(iii) A relationship of the form $\mathrm{B} = \mathrm{QAQ}^{-1}$ may also be regarded as a similarity transformation on A, since Q is the multiplicative inverse of Q^{-1}.

THEOREM

Two similar matrices, A and B, have the same eigenvalues. Furthermore, if the similarity transformation from A to B is $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}$, then the eigenvectors, X and Y, of A and B respectively are related by the equation

$$\mathrm{Y} = \mathrm{P}^{-1}\mathrm{X}.$$

Proof:

The eigenvalues, λ, and the eigenvectors, X, of A satisfy the relationship $\mathrm{AX} = \lambda\mathrm{X}$.

Hence,

$$\mathrm{P}^{-1}\mathrm{AX} = \lambda\mathrm{P}^{-1}X.$$

Secondly, using the fact that $\mathrm{PP}^{-1} = \mathrm{I}$, we have

$$\mathrm{P}^{-1}\mathrm{APP}^{-1}\mathrm{X} = \lambda \mathrm{P}^{-1}\mathrm{X},$$

which may be written

$$(\mathrm{P}^{-1}\mathrm{AP})(\mathrm{P}^{-1}\mathrm{X}) = \lambda(\mathrm{P}^{-1}\mathrm{X})$$

or

$$\mathrm{BY} = \lambda \mathrm{Y},$$

where $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}$ and $\mathrm{Y} = \mathrm{P}^{-1}\mathrm{X}$.

This shows that the eigenvalues of A are also the eigenvalues of B and that the eigenvectors of B are of the form $\mathrm{P}^{-1}\mathrm{X}$.

Reminders

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix};$$

and, in general, for a square matrix M,

$$\mathrm{M}^{-1} = \tfrac{1}{|\mathrm{M}|}\times \text{ the transpose of the cofactor matrix.}$$

9.8.3 EXERCISES

1. For the matrix,

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 0 \\ -2 & -1 & 1 \end{bmatrix},$$

and its multiplicative inverse, A^{-1}, determine the eigenvalues and a set of corresponding linearly independent normalised eigenvectors.

2. State the eigenvalues for the upper-triangular matrix

$$\begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

and, hence, obtain a set of linearly independent normalised eigenvectors for the matrix.

3. State the eigenvalues of the lower-triangular matrix

$$\begin{bmatrix} 6 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 1 & -10 \end{bmatrix}$$

and, hence, obtain a set of linearly independent normalised eigenvectors for the matrix.

4. Determine the eigenvalues and a set of corresponding linearly independent eigenvectors for the matrix $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}$, where

$$\mathrm{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \mathrm{P} = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}.$$

5. Determine the eigenvalues and a set of corresponding linearly independent eigenvectors for the matrix $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}$, where

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 0 \\ -2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{P} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -2 & 4 \\ 5 & 1 & 6 \end{bmatrix}.$$

6. Show that the matrix

$$\mathrm{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

is orthogonal and verify that its eigenvalues are either 1 or -1.

9.8.4 ANSWERS TO EXERCISES

1. The eigenvalues of A are 3, 2 and 1 with corresponding normalised eigenvectors,

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of A^{-1} are $\frac{1}{3}$, $\frac{1}{2}$ and 1, with corresponding normalised eigenvectors the same as for A.

2. The eigenvalues are 3, 2 and -1, with corresponding linearly independent normalised eigenvectors,

$$\frac{1}{\sqrt{17}}\begin{bmatrix}4\\-1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \text{and} \quad \frac{1}{3}\begin{bmatrix}2\\2\\-2\end{bmatrix}.$$

3. The eigenvalues are 6, 0 and -10, with corresponding linearly independent normalised eigenvectors,

$$\frac{1}{\sqrt{1305}}\begin{bmatrix}32\\16\\5\end{bmatrix}, \quad \frac{1}{\sqrt{101}}\begin{bmatrix}0\\10\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

4. The eigenvalues of B are the same as those of A, namely 0 and 7.

$$\text{Also,} \quad P^{-1} = \begin{bmatrix}2 & -7\\-1 & 4\end{bmatrix}.$$

Hence, a set of linearly independent eigenvectors for B is

$$\begin{bmatrix}-13\\7\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}-12\\7\end{bmatrix}.$$

5. The eigenvalues of B are the same as those of A which, from question 1, are 3, 2 and 1. Also,

$$P^{-1} = -\frac{1}{22}\begin{bmatrix}-16 & 9 & 2\\20 & -3 & -8\\10 & -7 & -4\end{bmatrix},$$

so that a set of linearly independent eigenvectors for B are

$$\begin{bmatrix}-17\\13\\12\end{bmatrix}, \quad \begin{bmatrix}-27\\31\\21\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}-1\\4\\2\end{bmatrix}.$$

6. $CC^T = I$ and the eigenvalues are 1 (repeated) and -1.