

“JUST THE MATHS”

UNIT NUMBER

9.7

MATRICES 7

(Linearly independent eigenvectors)

&

(Normalised eigenvectors)

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UNIT 9.7 - MATRICES 7

LINEARLY INDEPENDENT AND NORMALISED EIGENVECTORS

9.7.1 LINEARLY INDEPENDENT EIGENVECTORS

It is often useful to know if an $n \times n$ matrix, A , possesses a full set of n eigenvectors $X_1, X_2, X_3, \dots, X_n$, which are “**linearly independent**”.

That is, they are **not** connected by any relationship of the form

$$a_1X_1 + a_2X_2 + a_3X_3 + \dots \equiv 0,$$

where a_1, a_2, a_3, \dots are constants.

If the eigenvalues of A are distinct, it turns out that the eigenvectors are linearly independent; but, if any of the eigenvalues are repeated, further investigation may be necessary.

ILLUSTRATIONS

1. In Unit 9.6, it was shown that the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 8$ and $\lambda = -1$ (repeated), with corresponding eigenvectors,

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

where α is any **non-zero** scalar and

$$\beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

where β and γ are not both equal to zero at the same time.

The matrix, A , possesses a set of three linearly independent eigenvectors which may, conveniently, be chosen as

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

It is reasonably obvious that these are linearly independent, but a formal check would be to show that the matrix,

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

has rank 3.

2. It may be shown that the matrix,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix},$$

has eigenvalues $\lambda = 2$ (repeated) and $\lambda = 1$, with corresponding eigenvectors,

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \beta \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

where α and β are any non-zero numbers.

In this case, it is not possible to obtain a full set of three linearly independent eigenvectors.

9.7.2 NORMALISED EIGENVECTORS

It is sometimes convenient to use a set of “**normalised**” eigenvectors, which means that, for each eigenvector, the sum of the squares of its elements is equal to 1.

An eigenvector may be normalised if we multiply it by (plus or minus) the reciprocal of the square root of the sum of the squares of its elements.

ILLUSTRATIONS

1. A set of linearly independent normalised eigenvectors for the matrix,

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix},$$

is

$$\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

2. A set of linearly independent normalised eigenvectors for the matrix,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix},$$

is

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

9.7.3 EXERCISES

1. Determine the eigenvalues and a set of linearly independent normalised eigenvectors for the following 2×2 matrices:

$$(a) \begin{bmatrix} 17 & -6 \\ 45 & -16 \end{bmatrix}, \quad (b) \begin{bmatrix} 5 & -2 \\ 7 & -4 \end{bmatrix}, \quad (c) \begin{bmatrix} 16 & -8 \\ 24 & -12 \end{bmatrix}.$$

2. Determine the eigenvalues and a set of linearly independent normalised eigenvectors for the following 3×3 matrices:

(a)

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix};$$

(b)

$$\begin{bmatrix} 7 & 0 & 4 \\ 7 & 0 & 4 \\ 0 & 0 & 11 \end{bmatrix};$$

(c)

$$\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix};$$

(d)

$$\begin{bmatrix} -2 & 0 & -14 \\ -7 & 5 & -14 \\ 0 & 0 & 5 \end{bmatrix};$$

(e)

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}.$$

9.7.4 ANSWERS TO EXERCISES

1. (a) The eigenvalues are 2 and -1 .

A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{29}} \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

(b) The eigenvalues are 3 and -2 .

A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{53}} \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

(c) The eigenvalues are 4 and 0.

A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

2. (a) The eigenvalues are 1, 0 and -2 .

A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{230}} \begin{bmatrix} 10 \\ 3 \\ -11 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{74}} \begin{bmatrix} 4 \\ 3 \\ -7 \end{bmatrix}.$$

- (b) The eigenvalues are 11, 7 and 0.

A set of linearly independent normalised eigenvectors are

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

- (c) The eigenvalues are 2 (repeated) and -2 .

A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{66}} \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}.$$

- (d) The eigenvalues are 5 (repeated) and -2 .

A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

- (e) The eigenvalues are 4 (repeated) and 3.

A set of linearly independent normalised eigenvectors is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$