

**“JUST THE MATHS”**

**UNIT NUMBER**

**9.6**

**MATRICES 6**  
**(Eigenvalues and eigenvectors)**

by

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## UNIT 9.6 - MATRICES 6

### EIGENVALUES AND EIGENVECTORS

#### 9.6.1 THE STATEMENT OF THE PROBLEM

Suppose  $A$  is any square matrix, and let  $X$  be a column vector with the same number of rows as there are columns in  $A$ . For example, if  $A$  is of order  $m \times m$ , then  $X$  must be of order  $m \times 1$  and  $AX$  will also be of order  $m \times 1$ .

We ask the question:

“Is it ever possible that  $AX$  can be just a scalar multiple of  $X$  ?”

We exclude the case when the elements of  $X$  are all zero since, in a practical application, these elements will usually be the components of an actual vector quantity; and, if they are all zero, the direction of the vector will be indeterminate.

#### ILLUSTRATIONS

1.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix}.$$

2.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

#### The formal statement of the problem

For a given square matrix,  $A$ , we investigate the existence of column vectors,  $X$ , such that

$$AX = \lambda X$$

for some scalar quantity  $\lambda$ . Each such column vector is called an “**eigenvector**” of the matrix,  $A$ ; and each corresponding value of  $\lambda$  is called an “**eigenvalue**” of the matrix,  $A$ .

**Notes:**

(i) The German word “eigen” (meaning “hidden”) gives rise to the above names, but other alternatives are “latent values and latent vectors” or “characteristic values and characteristic vectors”.

(ii) In the discussion which follows,  $A$  will be, mostly, a matrix of order  $3 \times 3$ , but the ideas involved will apply to square matrices of other orders also (see Example 1 and Exercises 9.6.3, question 1).

**9.6.2 THE SOLUTION OF THE PROBLEM**

Assuming that

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the matrix equation,  $AX = \lambda X$ , may be written out fully in the following form:

$$\begin{aligned} a_1x + b_1y + c_1z &= \lambda x, \\ a_2x + b_2y + c_2z &= \lambda y, \\ a_3x + b_3y + c_3z &= \lambda z; \end{aligned}$$

or, on rearrangement,

$$\begin{aligned} (a_1 - \lambda)x + b_1y + c_1z &= 0, \\ a_2x + (b_2 - \lambda)y + c_2z &= 0, \\ a_3x + b_3y + (c_3 - \lambda)z &= 0, \end{aligned}$$

which is a set of homogeneous linear equations in  $x$ ,  $y$  and  $z$  and may be written, for short, in the form

$$(A - \lambda I)X = [0],$$

where  $I$  denotes the identity matrix of order  $3 \times 3$ .

From the results of Unit 7.4, the condition that the three homogeneous linear equations have a solution other than  $x = 0$ ,  $y = 0$ ,  $z = 0$  is

$$|A - \lambda I| = \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0.$$

This equation will give, on expansion, a cubic equation in  $\lambda$  called the “**characteristic equation**” of  $A$ . Its left-hand side is called the “**characteristic polynomial**” of  $A$ .

The characteristic equation of a  $3 \times 3$  matrix, being a cubic equation, will (in general) have three solutions.

## EXAMPLES

1. Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}.$$

### Solution

#### (a) The eigenvalues

The characteristic equation is given by

$$0 = |A - \lambda I| = \begin{vmatrix} 2 - \lambda & 4 \\ 5 & 3 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda - 14 = (\lambda + 2)(\lambda - 7).$$

The eigenvalues are therefore  $\lambda = -2$  and  $\lambda = 7$ .

#### (b) The eigenvectors

##### Case 1. $\lambda = -2$

We require to solve the equation  $x + y = 0$ ,

giving  $x : y = -1 : 1$  and a corresponding eigenvector

$$X = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where  $\alpha$  is any **non-zero** scalar.

##### Case 2. $\lambda = 7$

We require to solve the equation  $5x - 4y = 0$ ,

giving  $x : y = 4 : 5$  and a corresponding eigenvector

$$X = \beta \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

where  $\beta$  is any **non-zero** scalar.

2. Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

### Solution

#### (a) The eigenvalues

The characteristic equation is given by

$$0 = |A - \lambda I| = \begin{vmatrix} 3 - \lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3 - \lambda \end{vmatrix}.$$

Direct expansion of the determinant gives the equation

$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0,$$

which will factorise into

$$(1 + \lambda)^2(8 - \lambda) = 0.$$

#### Note:

Students who have studied row and column operations for determinants (see Unit 7.3) may obtain this by simplifying the determinant first. (One way is to subtract the third column from the first column and then add the third row to the first row).

The eigenvalues are therefore  $\lambda = -1$ (repeated) and  $\lambda = 8$ .

#### (b) The eigenvectors

**Case 1.**  $\lambda = 8$

We require to solve the homogeneous equations

$$\begin{aligned} -5x + 2y + 4z &= 0, \\ 2x - 8y + 2z &= 0, \\ 4x + 2y - 5z &= 0. \end{aligned}$$

Eliminating  $x$  from the second and third equations gives  $18y - 9z = 0$ .

Eliminating  $y$  from the second and third equations gives  $18x - 18z = 0$ .

Since  $z$  appears twice in the two statements, we may try letting  $z = 1$  to give  $y = \frac{1}{2}$  and  $x = 1$ .

Hence,

$$x : y : z = 1 : \frac{1}{2} : 1 = 2 : 1 : 2$$

The eigenvectors corresponding to  $\lambda = 8$  are thus given by

$$X = \alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

where  $\alpha$  is any **non-zero** scalar.

**Case 2.**  $\lambda = -1$

This time, we require to solve the homogeneous equations

$$\begin{aligned} 4x + 2y + 4z &= 0, \\ 2x + y + 2z &= 0, \\ 4x + 2y + 4z &= 0, \end{aligned}$$

which, of course, are all the same equation,  $2x + y + 2z = 0$ .

Hence, two of the variables may be chosen at random (say  $y = \beta$  and  $z = \gamma$ ), then the third variable may be expressed in terms of them; (in this case  $x = -\frac{1}{2}\beta - \gamma$ ).

However, a neater technique is to obtain, first, a pair of independent particular solutions by setting one pair of the variables at 1 and 0, in both orders. For example  $y = 1$  and  $z = 0$  gives  $x = -\frac{1}{2}$  while  $y = 0$  and  $z = 1$  gives  $x = -1$ .

The general solution (in which  $y$  and  $z$  are randomly chosen as  $\beta$  and  $\gamma$  respectively) is then given by

$$X = \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

where  $\beta$  and  $\gamma$  are **not both equal to zero** at the same time.

**Notes:**

(i) Similar results in Case 2 could be obtained by choosing **different** pairs of the three variables at random.

(ii) Other special cases arise if the three homogeneous equations reduce to a single equation in which one or even two of the variables is absent.

For example, if they reduced to  $y = 0$ , the corresponding eigenvectors could be given by

$$X = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

allowing  $x = \alpha$  and  $z = \gamma$  to be chosen at random, assuming that  $\alpha$  and  $\gamma$  are not both zero simultaneously.

Alternatively, if the homogeneous equations reduced to  $3x + 5z = 0$ , then the corresponding eigenvectors could be given by

$$X = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

allowing  $x = \alpha$  and  $y = \beta$  to be chosen at random, assuming that  $\alpha$  and  $\beta$  are not both zero simultaneously.

**9.6.3 EXERCISES**

Determine the eigenvalues and eigenvectors of the following matrices:

1.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

2.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

3.

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

4.

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

5.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

6.

$$\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

7.

$$\begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

8.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

9.

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 0 & 2 \end{bmatrix}.$$

10.

$$\begin{bmatrix} 4 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 3 \end{bmatrix}.$$

### 9.6.4 ANSWERS TO EXERCISES

The eigenvectors will be written as  $X^T$  instead of  $X$ , in order to avoid unnecessary waste of space.

- $\lambda = 0$ , giving  $X^T = \alpha[-3 \ 1]$ ,  
where  $\alpha$  is any non-zero scalar;  
 $\lambda = 7$ , giving  $X^T = \beta[1 \ 2]$ ,  
where  $\beta$  is any non-zero scalar.
- $\lambda = 1$ , giving  $X^T = \alpha[1 \ -1 \ 0]$ ,  
where  $\alpha$  is any non-zero scalar;  
 $\lambda = 2$ , giving  $X^T = \beta[-2 \ 1 \ 2]$ ,  
where  $\beta$  is any non-zero scalar;  
 $\lambda = 3$ , giving  $X^T = \gamma[-1 \ 1 \ 2]$ ,  
where  $\gamma$  is any non-zero scalar.
- $\lambda = 1$ , giving  $X^T = \alpha[3 \ 2 \ 1]$ ,  
where  $\alpha$  is any non-zero scalar;  
 $\lambda = -1$ , giving  $X^T = \beta[1 \ 0 \ 1]$ ,  
where  $\beta$  is any non-zero scalar;  
 $\lambda = 2$ , giving  $X^T = \gamma[1 \ 3 \ 1]$ ,  
where  $\gamma$  is any non-zero scalar.
- $\lambda = 1$ , giving  $X^T = \alpha[1 \ 1 \ -1]$ ,  
where  $\alpha$  is any non-zero scalar;  
 $\lambda = 2$  (repeated), giving  $X^T = \beta[2 \ 1 \ 0]$ ,  
where  $\beta$  is any non-zero scalar.
- $\lambda = 1$  (repeated), giving  $X^T = \alpha[1 \ 1 \ 1]$ ,  
where  $\alpha$  is any non-zero scalar.
- $\lambda = 0$ , giving  $X^T = \alpha[-4 \ 1 \ 0]$ ,  
where  $\alpha$  is any non-zero scalar;  
 $\lambda = 2$ , giving  $X^T = \beta[-2 \ 1 \ 0]$ ,  
where  $\beta$  is any non-zero scalar;  
 $\lambda = -1$ , giving  $X^T = \gamma[28 \ -8 \ 3]$ ,  
where  $\gamma$  is any non-zero scalar.

7.  $\lambda = 1$ , giving  $X^T = \alpha[-12 \ 4 \ 1]$ ,  
where  $\alpha$  is any non-zero scalar;  
 $\lambda = 0$  (repeated), giving  $X^T = \beta[-3 \ 1 \ 0] + \gamma[-4 \ 0 \ 1]$ ,  
where  $\beta$  and  $\gamma$  are any scalar numbers which are not simultaneously zero.
8.  $\lambda = 5$ , giving  $X^T = \alpha[1 \ 1 \ 1]$ ,  
where  $\alpha$  is any non-zero scalar;  
 $\lambda = 1$  (repeated), giving  $X^T = \beta[-2 \ 1 \ 0] + \gamma[-1 \ 0 \ 1]$ ,  
where  $\beta$  and  $\gamma$  are any scalar numbers which are not simultaneously zero.
9.  $\lambda = 2$  (repeated), giving  $X^T = \beta[0 \ 1 \ 0] + \gamma[0 \ 0 \ 1]$   
where  $\beta$  and  $\gamma$  are any scalar numbers which are not simultaneously zero.
10.  $\lambda = 2$ , giving  $X^T = \alpha[1 \ 1 \ 1]$ ,  
where  $\alpha$  is any non-zero number;  
 $\lambda = 3$ , giving  $X^T = \alpha[1 \ \frac{1}{2} \ 0] + \gamma[0 \ 0 \ 1]$ ,  
where  $\alpha$  and  $\gamma$  are any scalar numbers which are not simultaneously zero.