“JUST THE MATHS”

UNIT NUMBER

9.4

MATRICES 4
(Row operations)

by

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UNIT 9.4 - MATRICES 4 - ROW OPERATIONS

9.4.1 MATRIX INVERSES BY ROW OPERATIONS

In this section, we shall examine an alternative method for finding the multiplicative inverse of a matrix but the techniques introduced will lead on to other aspects of solving simultaneous linear equations not discussed in earlier units.

DEFINITION
An **"elementary row operation"** on a matrix is any one of the following three possibilities:

(a) The interchange of two rows;
(b) The multiplication of the elements in any row by a non-zero number;
(c) The addition of the elements in any row to the corresponding elements in another row.

Notes:
(i) Elementary row operations are essentially the same kind of operations as those used in the solution of a set of simultaneous linear equations by the method of elimination; but here we shall be considering sets of coefficients in the form of matrices rather than complete sets of equations.

(ii) Elementary row operations of types (b) and (c) imply that the elements in any row may be **subtracted** from the corresponding elements in another row and, more generally, multiples of the elements in any row may be added to or subtracted from the corresponding elements in another row.

RESULT 1.
To perform an elementary row operation on a matrix **algebraically**, we may pre-multiply the matrix by an identity matrix on which the same elementary row operation has been already performed. For example, in the matrix

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix},$$

suppose we wished to subtract twice the third row from the second row. It is easy enough to carry this out by inspection, but could also be regarded as the succession of two elementary row operations as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} . \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ 2a_3 & 2b_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 - 2a_3 & b_2 - 2b_3 \\ a_3 & b_3 \end{bmatrix}.$$

DEFINITION

An **"elementary matrix"** is a matrix obtained from an identity matrix by performing upon it one elementary row operation.

RESULT 2.

If a certain sequence of elementary row operations converts a given square matrix, M, into the corresponding identity matrix, then the same sequence of elementary row operations in the same order will convert the identity matrix into M^{-1}.

Proof:

Suppose that

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.M = \text{I}$$

where $E_1, E_2, E_3, E_4, ..., E_{n-1}, En$ are elementary matrices.

Then, by post-multiplying both sides with M^{-1}, we obtain

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.\text{M}.\text{M}^{-1} = \text{I}.\text{M}^{-1}.$$

In other words,

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.\text{I} = \text{M}^{-1},$$

which proves the result.

EXAMPLES

1. Use elementary row operations to determine the inverse of the matrix

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}.$$

Solution

First we write down the given matrix side-by-side with the corresponding identity matrix in the following format:

$$\begin{bmatrix} 3 & 7 & \vdots & 1 & 0 \\ 2 & 5 & \vdots & 0 & 1 \end{bmatrix}.$$

Secondly, we try to arrange that the first element in the first column of this arrangement is 1; and this can be carried out by subtracting the second row from the first row. (Instruction: $R_1 \to R_1 - R_2$).

$$\begin{bmatrix} 1 & 2 & \vdots & 1 & -1 \\ 2 & 5 & \vdots & 0 & 1 \end{bmatrix}.$$

Thirdly, we try to convert the first column of the display into the first column of the identity matrix; and this can be carried out by subtracting twice the first row from the second row.
(Instruction: $R_2 \to R_2 - 2R_1$).

$$\begin{bmatrix} 1 & 2 & \vdots & 1 & -1 \\ 0 & 1 & \vdots & -2 & 3 \end{bmatrix}.$$

Lastly, we try to convert the second column of the display into the second column of the identity matrix: and this can be carried out by subtracting twice the second row from the first row.
(Instruction: $R_1 \to R_1 - 2R_2$).

$$\begin{bmatrix} 1 & 0 & \vdots & 5 & -7 \\ 0 & 1 & \vdots & -2 & 3 \end{bmatrix}.$$

The inverse matrix is therefore

$$\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix},$$

as we would have obtained by the cofactor method.

Notes:

(i) The technique used for a 2×2 matrix applies to square matrices of all orders with appropriate modifications.

(ii) The idea of the method is to obtain, in chronological order, the columns of the identity matrix from the columns of the given matrix. We do this by using elementary row operations to convert each diagonal element in the given matrix to 1 and then using multiples of 1 to reduce the remaining elements in the same column to zero.

(iii) Once any row has been used to reduce elements to zero, that row must not be used again as the operator; otherwise the zeros obtained may change to other values.

2. Use elementary row operations to show that the matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

has no inverse.

Solution

We set up the scheme in the following format:

$$\begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 2 & 6 & \vdots & 0 & 1 \end{bmatrix}.$$

Then, we proceed according to the instructions indicated:

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & -1 & 1 \end{bmatrix}.$$

There is no way now of continuing to convert the given matrix into the identity matrix; hence, there is no inverse.

3. Use elementary row operations to determine the inverse of the matrix

$$\begin{bmatrix} 4 & 1 & 6 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}.$$

Solution

We set up the scheme in the following format:

$$\begin{bmatrix} 4 & 1 & 6 & \vdots & 1 & 0 & 0 \\ 2 & 1 & 3 & \vdots & 0 & 1 & 0 \\ 3 & 2 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Then, we proceed according to the instructions indicated:

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 1 & 0 & 1 \\ 2 & 1 & 3 & \vdots & 0 & 1 & 0 \\ 3 & 2 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix};$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 1 & 0 & -1 \\ 0 & 3 & 1 & \vdots & -2 & 1 & 2 \\ 0 & 5 & 2 & \vdots & -3 & 0 & 4 \end{bmatrix};$$

$R_2 \to 2R_2 - R_3$

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 1 & 0 & -1 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 5 & 2 & \vdots & -3 & 0 & 4 \end{bmatrix};$$

$R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 2 & -1 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 2 & \vdots & 2 & -10 & 4 \end{bmatrix};$$

$R_3 \to R_3 \times \frac{1}{2}$

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 2 & -1 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 1 & \vdots & 1 & -5 & 2 \end{bmatrix};$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -1 & 7 & -3 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 5 & 2 \end{bmatrix}.$$

The required inverse matrix is therefore

$$\begin{bmatrix} -1 & 7 & -3 \\ -1 & 2 & 0 \\ 1 & -5 & 2 \end{bmatrix}.$$

9.4.2 GAUSSIAN ELIMINATION - THE ELEMENTARY VERSION

Elementary row operations can also be conveniently used in another method of solving simultaneous linear equations which relates closely again to the elimination method sometimes encountered in courses which do not include matrices. The method will be introduced through the case of three equations in three unkowns, but may be applied in other cases as well.

Suppose a set of simultaneous linear equations in the variables x, y and z appeared in the special form:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ b_2y + c_2z &= k_2, \\ c_3z &= k_3. \end{aligned}$$

Then it is very simple to solve the equations by first obtaining z from the third equation, substituting its value into the second equation in order to find y, then substituting for both y and z in the first equation in order to find x.

The method of Gaussian Elimination reduces any set of linear equations to this triangular form by adding or subtracting suitable multiples of pairs of the equations; but the method is more conveniently laid out in a tabular form using only the coefficients of the variables and the constant terms. We illustrate with an example.

EXAMPLE
Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution
For the simplest arithmetic, we try to arrange that the first coefficient in the first equation is 1. In the current example, we could interchange the first two equations.

$\boxed{1}$	-2	-1	2
2	1	1	3
3	-1	1	8

This format is known as an **"augmented matrix"**. The matrix of coefficients has been augmented by the matrix of constant terms.

Using the notation of the previous section, we apply the instructions $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ giving a new table, namely

$$\begin{array}{cccc|c} 1 & -2 & -1 & 2 \\ 0 & \boxed{5} & 3 & -1 \\ 0 & 5 & 4 & 2 \end{array}$$

Next, we apply the instruction $R_3 \rightarrow R_3 - R_2$ giving

$$\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & 1 & 3 \end{array}$$

The numbers enclosed in the boxes are called the **"pivot elements"** and are used to reduce to zero the elements below them in the same column.

The final table above provides a new set of equations, equivalent to the original, namely

$$\begin{aligned} x - 2y - z &= 2, \\ 5y + 3z &= -1, \\ z &= 3. \end{aligned}$$

Hence, $\boxed{z = 3, \quad y = -2, \quad x = 1}$.

INSERTING A CHECK COLUMN

In the above example, the numbers were fairly simple, giving little scope for careless mistakes. However, with large numbers of equations, often involving awkward decimal quantities, the margin for error is greatly increased.

As a check on the arithmetic at each stage, we deliberately introduce some **additional** arithmetic which has to remain consistent with the calculations already being carried out. The method is to add together the numbers in each row in order to produce an extra column; each row operation is then performed on the extended rows with the result that, in the new table, the final column should still be the sum of the numbers to the left of it.

The working for our previous example would be as follows:

$$\begin{array}{ccc|c|c} \boxed{1} & -2 & -1 & 2 & 0 \\ 2 & 1 & 1 & 3 & 7 \\ 3 & -1 & 1 & 8 & 11 \end{array}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$\begin{array}{ccc|c|c} 1 & -2 & -1 & 2 & 0 \\ 0 & \boxed{5} & 3 & -1 & 7 \\ 0 & 5 & 4 & 2 & 11 \end{array}$$

$R_3 \to R_3 - R_2$

$$\begin{array}{ccc|c|c} 1 & -2 & -1 & 2 & 0 \\ 0 & 5 & 3 & -1 & 7 \\ 0 & 0 & 1 & 3 & 4 \end{array}$$

9.4.4 EXERCISES

1. Use elementary row operations to determine (where possible) the inverses of the following matrices:

   (a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$;

   (b) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$;

   (c) $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$;

   (d) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -7 \\ 3 & 11 & 3 \end{bmatrix}$;

   (e) $\begin{bmatrix} 0 & 3 & -2 \\ 1 & -1 & 5 \\ 1 & 5 & 1 \end{bmatrix}$;

   (f) $\begin{bmatrix} 2 & -3 & 4 \\ -1 & 3 & -4 \\ 1 & 0 & 2 \end{bmatrix}$;

   (g) $\begin{bmatrix} -8 & -7 & -6 & -5 \\ -4 & -3 & -2 & -1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$.

2. Use Gaussian Elimination (with a check column) to solve the following sets of simultaneous linear equations:

   (a)

$$\begin{aligned} x + 3y &= -8, \\ 5x - 2y &= 11; \end{aligned}$$

   (b)

$$\begin{aligned} x + 2y &= -2, \\ 5x - 4y &= 3; \end{aligned}$$

   (c)

$$\begin{aligned} 2x - y + z &= 7, \\ 3x + y - 5z &= 13, \\ x + y + z &= 5; \end{aligned}$$

   (d)

$$\begin{aligned} 3x + 2y - 2z &= 16, \\ 4x + 3y + 3z &= 2, \\ 2x - y + z &= -1. \end{aligned}$$

9.4.5 ANSWERS TO EXERCISES

1. (a) $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$;

   (b) $\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$;

   (c) $\begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{2} & -1 \end{bmatrix}$;

   (d) $\begin{bmatrix} 77 & -17 & -14 \\ -27 & 6 & 5 \\ 22 & -5 & -4 \end{bmatrix}$;

   (e) There is no inverse;

   (f) $\begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$;

   (g) There is no inverse.

2. (a) $x = 1$ and $y = -3$;

   (b) $x = -\frac{1}{7}$ and $y = -\frac{13}{14}$;

   (c) $x = 4$, $y = 1$ and $z = 0$;

   (d) $x = 2$, $y = \frac{3}{2}$ and $z = -\frac{7}{2}$.