“JUST THE MATHS”

UNIT NUMBER

8.4

VECTORS 4
(Triple products)

by

A.J.Hobson

UNIT 8.4 - VECTORS 4

TRIPLE PRODUCTS

INTRODUCTION

Once the ideas of scalar (dot) product and vector (cross) product for two vectors has been introduced, it is then possible to consider certain products of three or more vectors where, in some cases, there may be a mixture of scalar and vector products.

8.4.1 THE TRIPLE SCALAR PRODUCT

DEFINITION 1

Given three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, expressions such as

$$\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c}), \quad \underline{b} \bullet (\underline{c} \textbf{ x } \underline{a}), \quad \underline{c} \bullet (\underline{a} \textbf{ x } \underline{b})$$

or

$$(\underline{a} \textbf{ x } \underline{b}) \bullet \underline{c}, \quad (\underline{b} \textbf{ x } \underline{c}) \bullet \underline{a}, \quad (\underline{c} \textbf{ x } \underline{a}) \bullet \underline{b}$$

are called **"triple scalar products"** because their results are all scalar quantities. Strictly speaking, the brackets are not necessary because there is no ambiguity without them; that is, it is not possible to form the vector product of a vector with the result of a scalar product.

In the work which follows, we shall take $\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c})$ as the typical triple scalar product.

The formula for a triple scalar product

Suppose that

$$\underline{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \underline{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \text{ and } \underline{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

Then,

$$\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From the basic formula for scalar product, this becomes

$$\underline{\text{a}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c3 \end{vmatrix}.$$

Notes:

(i) From Unit 7.3, if two rows of a determinant are interchanged, the determinant remains unchanged in numerical value but is altered in sign.

Hence,

$$\underline{\text{a}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{c}}) = -\underline{\text{a}} \bullet (\underline{\text{c}} \textbf{ x } \underline{\text{b}}) = \underline{\text{c}} \bullet (\underline{\text{a}} \textbf{ x } \underline{\text{b}}) = -\underline{\text{c}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{a}}) = \underline{\text{b}} \bullet (\underline{\text{c}} \textbf{ x } \underline{\text{a}}) = -\underline{\text{b}} \bullet (\underline{\text{a}} \textbf{ x } \underline{\text{c}}).$$

In other words, the **"cyclic permutations"** of $\underline{\text{a}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{c}})$ are all equal in numerical value and in sign, while the remaining permutations are equal to $\underline{\text{a}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{c}})$ in numerical value, but opposite in sign.

(ii) The triple scalar product, $\underline{\text{a}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{c}})$, is often denoted by $[\underline{\text{a}}, \underline{\text{b}}, \underline{\text{c}}]$.

EXAMPLE

Evaluate the triple scalar product, $\underline{\text{a}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{c}})$, in the case when

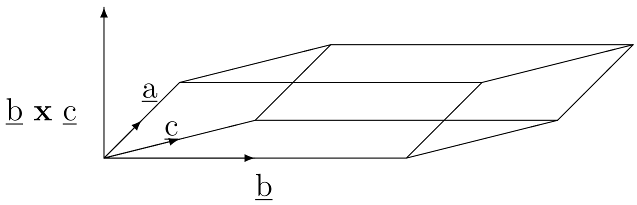
$$\underline{\text{a}} = 2\mathbf{i} + \mathbf{k}, \quad \underline{\text{b}} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \underline{\text{c}} = -\mathbf{i} + \mathbf{j}$$

Solution

$$\underline{\text{a}} \bullet (\underline{\text{b}} \textbf{ x } \underline{\text{c}}) = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 2.(-2) - 0.(2) + 1.(2) = -2.$$

A geometrical application of the triple scalar product

Suppose that the three vectors $\underline{\text{a}}$, $\underline{\text{b}}$ and $\underline{\text{c}}$ lie along three adjacent edges of a parallelepiped (correct pronunciation, *"parallel-epi-ped"*) as shown in the following diagram:



The area of the base of the parallelepiped, from the geometrical properties of vector products, is the **magnitude** of the vector, $\underline{b}$ **x** $\underline{c}$, which is perpendicular to the base.

The perpendicular height of the parallelepiped is the projection of the vector $\underline{a}$ onto the vector $\underline{b}$ **x** $\underline{c}$; that is,

$$\frac{\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c})}{|\underline{b} \mathbf{x} \underline{c}|}.$$

Hence, since the volume, V, of the parallelepiped is equal to the area of the base times the perpendicular height, we conclude that

$$V = \underline{a} \bullet (\underline{b} \mathbf{x} \underline{c}),$$

at least numerically, since the triple scalar product could turn out to be negative.

Note:
The above geometrical application also provides a condition that three given vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$ lie in the same plane; that is, they are **"coplanar"**.

The condition is that

$$\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c}) = 0,$$

since the three vectors would determine a parallelepiped whose volume is zero.

8.4.2 THE TRIPLE VECTOR PRODUCT

DEFINITION 2

If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three vectors, then the expression

$$\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c})$$

is called the **"triple vector product"** of $\underline{a}$ with $\underline{b}$ and $\underline{c}$.

Notes:

(i) The triple vector product is clearly a vector quantity.

(ii) The inclusion of the brackets in a triple vector product is important since it can be shown that, in general,

$$\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c}) \neq (\underline{a}\ \mathbf{x}\ \underline{b})\ \mathbf{x}\ \underline{c}.$$

For example, if the three vectors are considered as position vectors, with the origin as a common end-point, then $\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c})$ is perpendicular to both $\underline{a}$ and $\underline{b}\ \mathbf{x}\ \underline{c}$, the latter of which is already perpendicular to both $\underline{b}$ and $\underline{c}$. It therefore lies in the plane of $\underline{b}$ and $\underline{c}$.

Consequently, $(\underline{a}\ \mathbf{x}\ \underline{b})\ \mathbf{x}\ \underline{c}$, which is the same as $-\underline{c}\ \mathbf{x}\ (\underline{a}\ \mathbf{x}\ \underline{b})$, will lie in the plane of $\underline{a}$ and $\underline{b}$.

Hence it will, in general, be different from $\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c})$.

The formula for a triple vector product

Suppose that

$$\underline{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \underline{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \quad \text{and} \quad \underline{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

Then,

$$\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c}) = \underline{a}\ \mathbf{x} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix}.$$

The **i** component of this vector is equal to

$$a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) = b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3);$$

but, by adding and subtracting $a_1b_1c_1$, the right hand side can be rearranged in the form

$$(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1,$$

which is the **i** component of the vector $(\underline{\mathrm{a}} \bullet \underline{\mathrm{c}})\underline{\mathrm{b}} - (\underline{\mathrm{a}} \bullet \underline{\mathrm{b}})\underline{\mathrm{c}}$.

Similar expressions can be obtained for the **j** and **k** components and we may conclude that

$$\underline{\mathrm{a}} \ \mathbf{x} \ (\underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{c}}) = (\underline{\mathrm{a}} \bullet \underline{\mathrm{c}})\underline{\mathrm{b}} - (\underline{\mathrm{a}} \bullet \underline{\mathrm{b}})\underline{\mathrm{c}}.$$

EXAMPLE

Determine the triple vector product of $\underline{\mathrm{a}}$ with $\underline{\mathrm{b}}$ and $\underline{\mathrm{c}}$, where

$$\underline{\mathrm{a}} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \underline{\mathrm{b}} = -2\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad \underline{\mathrm{c}} = 3\mathbf{k}.$$

Solution

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{c}} = -3 \quad \text{and} \quad \underline{a} \bullet \underline{\mathrm{b}} = 4.$$

Hence,

$$\underline{\mathrm{a}} \ \mathbf{x} \ (\underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{c}}) = -3\underline{\mathrm{b}} - 4\underline{\mathrm{c}} = 6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k}.$$

8.4.3 EXERCISES

1. Evaluate the triple scalar product, $\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c})$, in the case when

$$\underline{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}, \quad \underline{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \underline{c} = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}.$$

2. Determine the volume of the parallelepiped with adjacent edges defined by the vectors

$$\underline{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \underline{b} = 2\mathbf{i} - \mathbf{j} \quad \text{and} \quad \underline{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

3. Determine the triple vector product of $\underline{a}$ with $\underline{b}$ and $\underline{c}$ in the cases where

   (a)

$$\underline{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \underline{b} = 2\mathbf{i} + \mathbf{j} \quad \text{and} \quad \underline{c} = \mathbf{i} + \mathbf{j} + \mathbf{k};$$

   (b)

$$\underline{a} = 4\mathbf{i} - \mathbf{k}, \quad \underline{b} = 3\mathbf{i} + 5\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{c} = \mathbf{i} - \mathbf{j} - \mathbf{k};$$

   (c)

$$\underline{a} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \underline{b} = 5\mathbf{i} \quad \text{and} \quad \underline{c} = -\mathbf{j} + 3\mathbf{k}.$$

4. Show that the following three vectors are coplanar:

$$\underline{a} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \underline{b} = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \underline{c} = -3\mathbf{i} + 12\mathbf{j} - 9\mathbf{k}.$$

5. Show that

$$[(\underline{a} + \underline{b}), (\underline{b} + \underline{c}), (\underline{c} + \underline{a})] = 2[\underline{a}, \underline{b}, \underline{c}].$$

6. Show that

$$(\underline{a} \textbf{ x } \underline{b}) \textbf{ x} (\underline{c} \textbf{ x } \underline{d}) = [\underline{a}, \underline{c}, \underline{d}]\underline{b} - [\underline{b}, \underline{c}, \underline{d}]\underline{a} = [\underline{a}, \underline{b}, \underline{d}]\underline{c} - [\underline{a}, \underline{b}, \underline{c}]\underline{d}.$$

7. Show that

$$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c}) + \underline{b} \textbf{ x } (\underline{c} \textbf{ x } \underline{a}) + \underline{c} \textbf{ x } (\underline{a} \textbf{ x } \underline{b}) = \mathbf{O}.$$

8. Show that

$$(\underline{a} \textbf{ x } \underline{b}) \bullet (\underline{c} \textbf{ x } \underline{d}) = \begin{vmatrix} \underline{a} \bullet \underline{c} & \underline{a} \bullet \underline{d} \\ \underline{b} \bullet \underline{c} & \underline{b} \bullet \underline{d} \end{vmatrix}.$$

8.4.4 ANSWERS TO EXERCISES

1.

$$20.$$

2.

$$8.$$

3. (a)

$$7\mathbf{i} + 4\mathbf{j} + \mathbf{k};$$

   (b)

$$2\mathbf{i} + 38\mathbf{j} + 8\mathbf{k};$$

   (c)

$$5\mathbf{i} + 20\mathbf{j} - 60\mathbf{k}.$$

4. Show that the triple scalar product is zero.
5. Remove all brackets and use the fact that a triple scalar product is zero when two of the vectors are the same.
6. Use the triple vector product formula.
7. Use the triple vector product formula.
8. Rearrange in the form

$$\underline{a} \bullet [\underline{b} \ \mathbf{x} \ (\underline{c} \ \mathbf{x} \ \underline{d})].$$