

**“JUST THE MATHS”**

**UNIT NUMBER**

**5.7**

**GEOMETRY 7**  
**(Conic sections - the ellipse)**

**by**

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<p><b>5.7.1 Introduction (the standard ellipse)</b> <b>5.7.2 A more general form for the equation of an ellipse</b> <b>5.7.2 Exercises</b> <b>5.7.3 Answers to exercises</b></p>
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## UNIT 5.7 - GEOMETRY 7

### CONIC SECTIONS - THE ELLIPSE

#### 5.7.1 INTRODUCTION

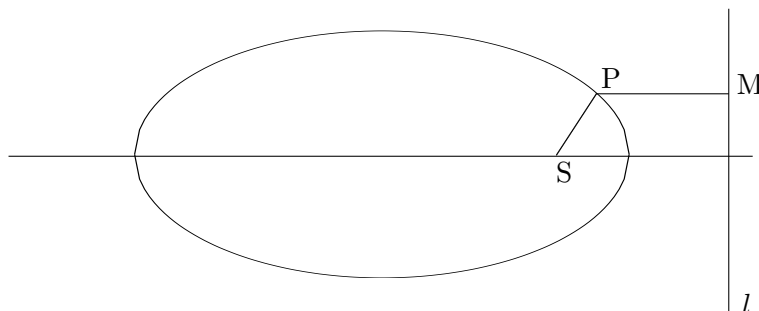
##### The Standard Form for the equation of an Ellipse

###### DEFINITION

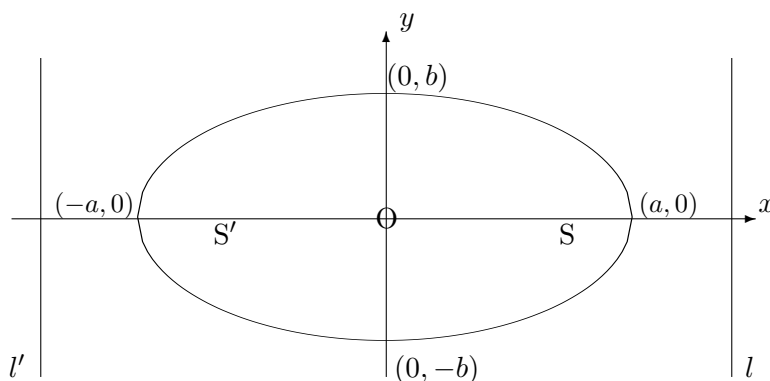
The Ellipse is the path traced out by (or “locus of”) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line,  $l$ , satisfy a relationship of the form

$$SP = \epsilon.PM,$$

where  $\epsilon < 1$  is a constant called the “eccentricity” of the ellipse. The fixed line,  $l$ , is called a “directrix” of the ellipse and the fixed point, S, is called a “focus” of the ellipse.



In fact, there are two foci and two directrices because the curve turns out to be symmetrical about a line parallel to  $l$  and the perpendicular line from S onto  $l$ . The diagram below illustrates two foci, S and S', together with two directrices,  $l$  and  $l'$ . The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a \cos \theta, \quad y = b \sin \theta.$$

The curve clearly intersects the  $x$ -axis at  $(\pm a, 0)$  and the  $y$ -axis at  $(0, \pm b)$ . Whichever is the larger of  $a$  and  $b$  defines the length of the “**semi-major axis**” and whichever is the smaller defines the length of the “**semi-minor axis**”.

For the sake of completeness, it may further be shown that the eccentricity,  $\epsilon$ , is obtainable from the formula

$$b^2 = a^2 (1 - \epsilon^2)$$

and, having done so, the foci lie at  $(\pm a\epsilon, 0)$  with directrices at  $x = \pm \frac{a}{\epsilon}$ . However, in these units, students will not normally be expected to determine the eccentricity, foci or directrices of an ellipse.

### 5.7.2 A MORE GENERAL FORM FOR THE EQUATION OF AN ELLIPSE

The equation of an ellipse, with centre  $(h, k)$  and axes of symmetry parallel to  $Ox$  and  $Oy$  respectively, is easily obtainable from the standard form of equation by a temporary change of origin to the point  $(h, k)$ . We obtain

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

with associated parametric equations

$$x = h + a \cos \theta, \quad y = k + b \sin \theta.$$

Ellipses will usually be encountered in the expanded form of the above cartesian equation and it will be necessary to complete the square in both the  $x$  terms and the  $y$  terms in order to locate the centre of the ellipse. The expanded form will be similar in appearance to that of a circle but the coefficients of  $x^2$  and  $y^2$ , though both of the same sign, will not be equal to each other.

#### EXAMPLE

Determine the co-ordinates of the centre and the lengths of the semi-axes of the ellipse whose equation is

$$3x^2 + y^2 + 12x - 2y + 1 = 0.$$

#### Solution

Completing the square in the  $x$  terms gives

$$3x^2 + 12x \equiv 3 [x^2 + 4x] \equiv 3 [(x + 2)^2 - 4] \equiv 3(x + 2)^2 - 12.$$

Completing the square in the  $y$  terms gives

$$y^2 - 2y \equiv (y - 1)^2 - 1.$$

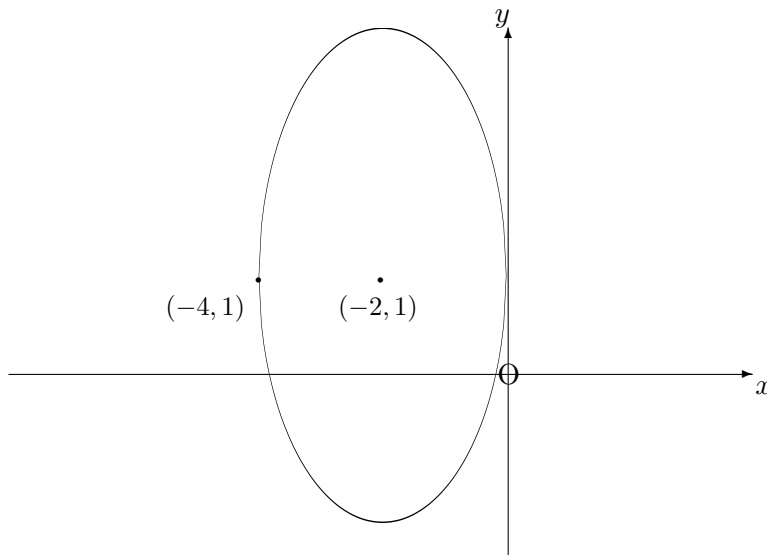
Hence, the equation of the ellipse becomes

$$3(x + 2)^2 + (y - 1)^2 = 12.$$

That is,

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{12} = 1.$$

The centre is thus located at the point  $(-2, 1)$  and the semi-axes have lengths  $a = 2$  and  $b = \sqrt{12}$ .



### 5.7.3 EXERCISES

1. For each of the following ellipses, determine the co-ordinates of the centre and give a sketch of the curve:

(a)

$$x^2 + 4y^2 - 4x - 8y + 4 = 0;$$

(b)

$$x^2 + 4y^2 + 16y + 12 = 0;$$

(c)

$$x^2 + 4y^2 + 6x - 8y + 9 = 0.$$

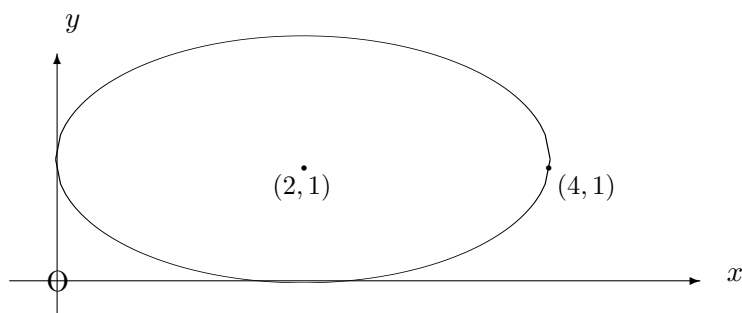
2. Determine the lengths of the semi-axes of the ellipse whose equation is

$$9x^2 + 25y^2 = 225$$

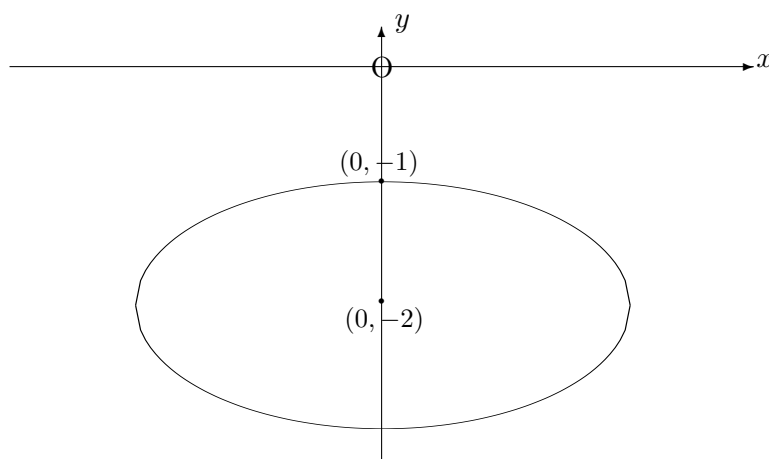
and write down also a pair of parametric equations for this ellipse.

### 5.7.4 ANSWERS TO EXERCISES

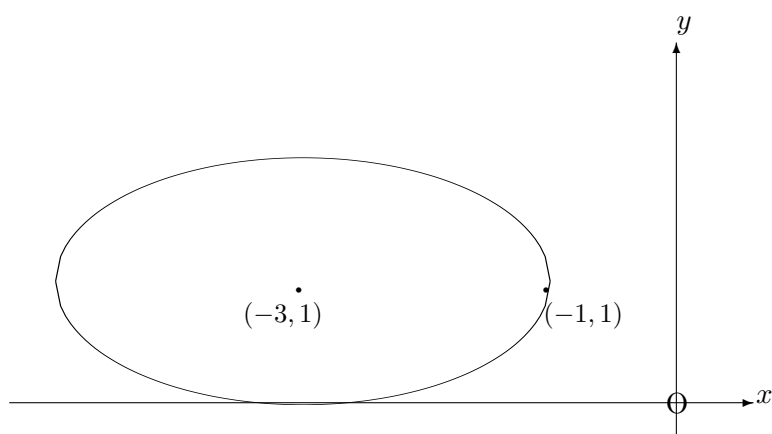
1. (a) Centre  $(2, 1)$  with  $a = 2$  and  $b = 1$ .



- (b) Centre  $(0, -2)$  with  $a = 2$  and  $b = 1$ .



- (c) Centre  $(-3, 1)$  with  $a = 2$  and  $b = 1$ .



2.  $a = 5$  and  $b = 3$ , giving the parametric equations  $x = 5 \cos \theta$ ,  $y = 3 \sin \theta$ .