"JUST THE MATHS"

UNIT NUMBER

5.4

GEOMETRY 4
(Elementary linear programming)

by

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UNIT 5.4 - GEOMETRY 4

ELEMENTARY LINEAR PROGRAMMING

5.4.1 FEASIBLE REGIONS

(i) The equation, $y = mx + c$, of a straight line is satisfied only by points which lie on the line. But it is useful to investigate the conditions under which a point with co-ordinates (x, y) may lie on one side of the line or the other.

(ii) For example, the inequality $y < mx + c$ is satisfied by points which lie **below** the line and the inequality $y > mx + c$ is satisfied by points which lie **above** the line.

(iii) Linear inequalities of the form $Ax+By+C < 0$ or $Ax+By+C > 0$ may be interpretted in the same way by converting, if necessary, to one of the forms in (ii).

(iv) Weak inequalities of the form $Ax + By + C \leq 0$ or $Ax + By + C \geq 0$ include the points which lie on the line itself as well as those lying on one side of it.

(v) Several simultaneous linear inequalties may be used to determine a region of the xy-plane throughout which all of the inequalities are satisfied. The region is called the **"feasible region"**.

EXAMPLES

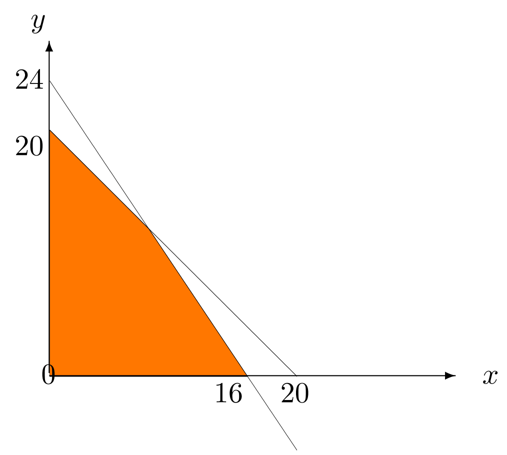
1. Determine the feasible region for the simultaneous inequalities

$$x \geq 0,\ \ y \geq 0,\ \ x + y \leq 20,\ \text{ and } 3x + 2y \leq 48$$

Solution

We require the points of the first quadrant which lie on or below the straight line $y = 20 - x$ and on or below the straight line $y = -\frac{3}{2}x + 16$.

The feasible region is shown as the shaded area in the following diagram:



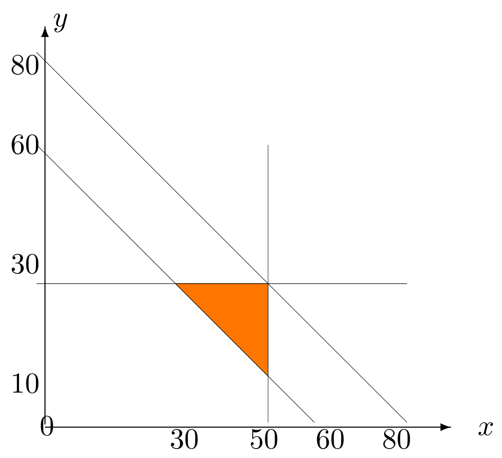
2. Determine the feasible region for the following simultaneous inequalities:

$$0 \leq x \leq 50, \ 0 \leq y \leq 30, \ x + y \leq 80, \ x + y \geq 60$$

Solution

We require the points which lie on or to the left of the straight line $x = 50$, on or below the straight line $y = 30$, on or below the straight line $y = 80 - x$ and on or above the straight line $y = 60 - x$.

The feasible region is shown as the shaded area in the following diagram:



5.4.2 OBJECTIVE FUNCTIONS

An important application of the feasible region discussed in the previous section is that of maximising (or minimising) a linear function of the form $px + qy$ subject to a set of simultaneous linear inequalities. Such a function is known as an **"objective function"**

Essentially, it is required that a straight line with gradient $-\frac{p}{q}$ is moved across the appropriate feasible region until it reaches the highest possible point of that region for a maximum value or the lowest possible point for a minimum value. This will imply that the straight line $px + qy = r$ is such that r is the optimum value required.

However, for convenience, it may be shown that the optimum value of the objective function always occurs at one of the corners of the feasible region so that we simply evaluate it at each corner and choose the maximum (or minimum) value.

EXAMPLES

1. A farmer wishes to buy a number of cows and sheep. Cows cost £18 each and sheep cost £12 each.

   The farmer has accommodation for not more than 20 animals, and cannot afford to pay more than £288.

   If he can reasonably expect to make a profit of £11 per cow and £9 per sheep, how many of each should he buy in order to make his total profit as large as possible ?

   Solution

   Suppose he needs to buy x cows and y sheep; then, his profit is the objective function $P \equiv 11x + 9y$.

   Also,

   $$x \geq 0,\ y \geq 0,\ x + y \leq 20,\ \text{and } 18x + 12y \leq 288 \text{ or } 3x + 2y \leq 48.$$

   Thus, we require to maximize $P \equiv 11x + 9y$ in the feasible region for the first example of the previous section.

   The corners of the region are the points $(0, 0)$, $(16, 0)$, $(0, 20)$ and $(8, 12)$, the last of these being the point of intersection of the two straight lines $x+y = 20$ and $3x+2y = 48$.

   The maximum value occurs at the point $(8, 12)$ and is equal to $88 + 108 = 196$. Hence, the farmer should buy 8 cows and 12 sheep.

2. A cement manufacturer has two depots, D_1 and D_2, which contain current stocks of 80 tons and 20 tons of cement respectively.

   Two customers C_1 and C_2 place orders for 50 and 30 tons respectively.

   The transport cost is £1 per ton, per mile and the distances, in miles, between D_1, D_2, C_1 and C_2 are given by the following table:

	C_1	C_2
D_1	40	30
D_2	10	20

   From which depots should the orders be dispatched in order to minimise the transport costs ?

   Solution

   Suppose that D_1 distributes x tons to C_1 and y tons to C_2; then D_2 must distribute $50 - x$ tons to C_1 and $30 - y$ tons to C_2.

   All quantities are positive and the following inequalties must be satisfied:

$$x \leq 50,\ \ y \leq 30,\ \ x + y \leq 80,\ \ 80 - (x + y) \leq 20 \text{ or } x + y \geq 60$$

   The total transport costs, T, are made up of $40x,\ 30y,\ 10(50 - x)$ and $20(30 - y)$.

   That is,

$$T \equiv 30x + 10y + 1100,$$

   and this is the objective function to be minimised.

   From the diagram in the second example of the previous section, we need to evaluate the objective function at the points $(30, 30),\ (50, 30)$ and $(50, 10)$.

   The minimum occurs, in fact, at the point $(30, 30)$ so that D_1 should send 30 tons to C_1 and 30 tons to C_2 while D_2 should send 20 tons to C_1 but 0 tons to C_2.

5.4.3 EXERCISES

1. Sketch, on separate diagrams, the regions of the xy-plane which correspond to the following inequalities (assuming that $x \geq 0$ and $y \geq 0$):

   (a)
   $$x + y \leq 6;$$
   (b)
   $$x + y \geq 4;$$
   (c)
   $$3x + y \geq 6;$$
   (d)
   $$x + 3y \geq 6.$$

2. Sketch the feasible region for which all the inequalities in question 1 are satisfied.

3. Maximise the objective function $5x + 7y$ subject to the simultaneous linear inequalities
   $$x \geq 0,\ y \geq 0,\ 3x + 2y \geq 6 \text{ and } x + y \leq 4.$$

4. A mine manager has contracts to supply, weekly,

   100 tons of grade 1 coal,
   700 tons of grade 2 coal,
   2000 tons of grade 3 coal,
   4500 tons of grade 4 coal.

   Two seams, A and B, are being worked at a cost of £4000 and £10,000, respectively, per shift, and the yield, in tons per shift, from each seam is given by the following table:

	Grade 1	Grade 2	Grade 3	Grade 4
A	200	100	200	400
B	100	100	500	1500

   How many shifts per week should each seam be worked, in order to fulfill the contracts most economically ?

5. A manufacturer employs 5 skilled and 10 semi-skilled workers to make an article in two qualities, standard and deluxe.

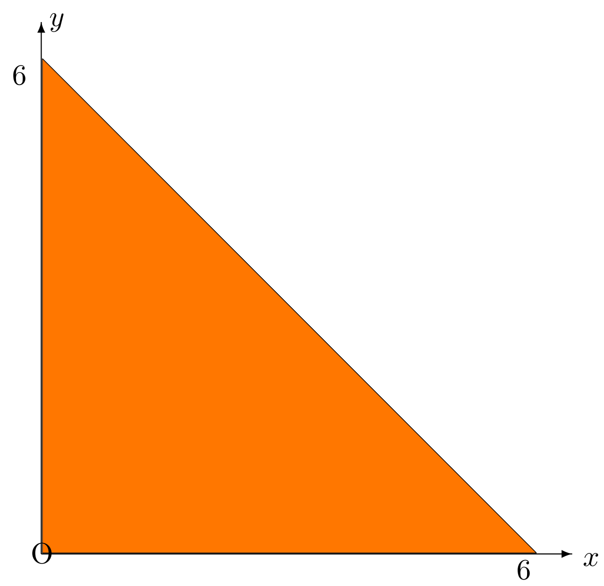
   The deluxe model requires 2 hour's work by skilled workers; the standard model requires 1 hour's work by skilled workers and 3 hour's work by semi-skilled workers.

   No worker works more than 8 hours per day and profit is £10 on the deluxe model and £8 on the standard model.

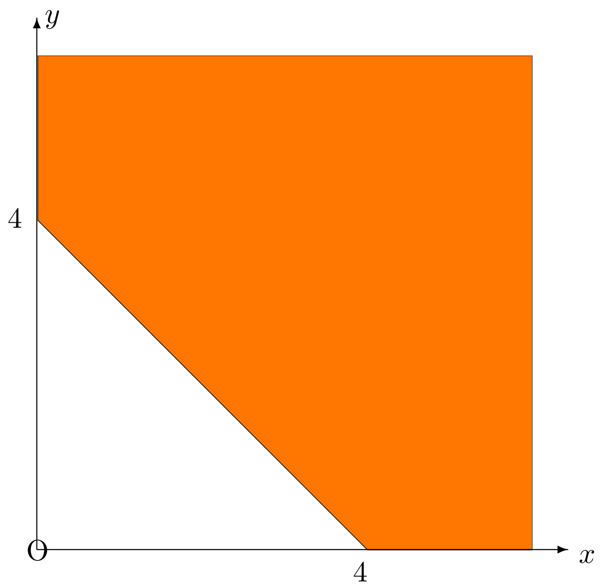
   How many of each type, per day, should be made in order to maximise profits ?

5.4.4 ANSWERS TO EXERCISES

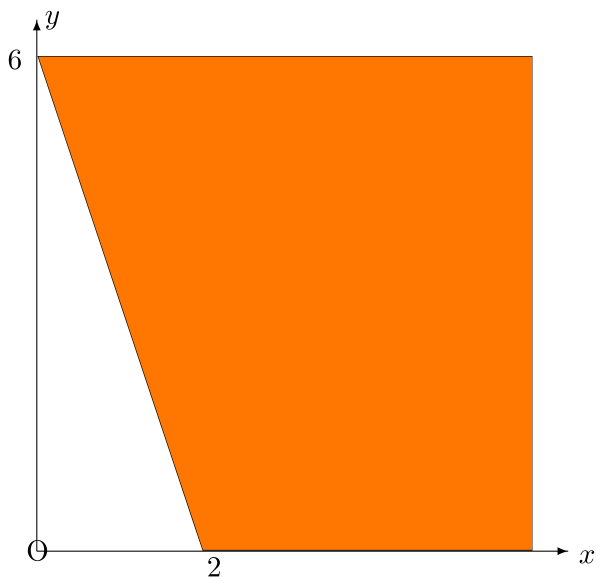
1. (a) The region is as follows:



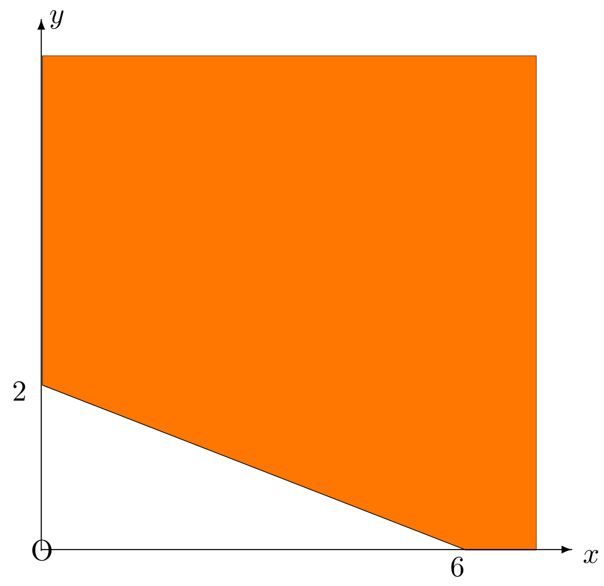
(b) The region is as follows:



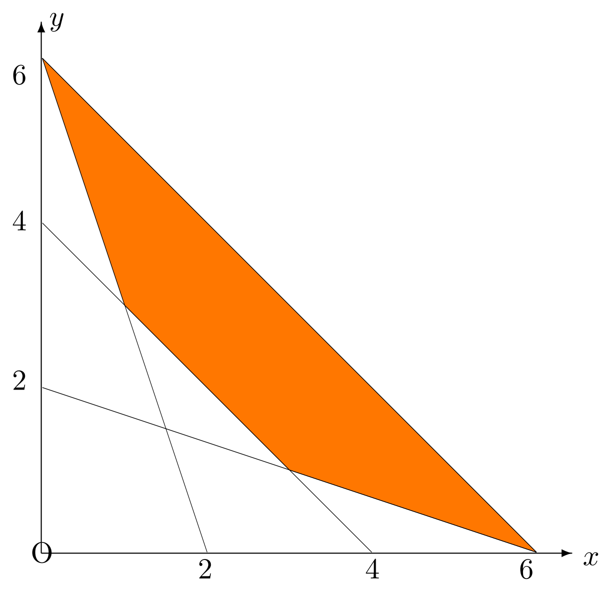
(c) The region is as follows:



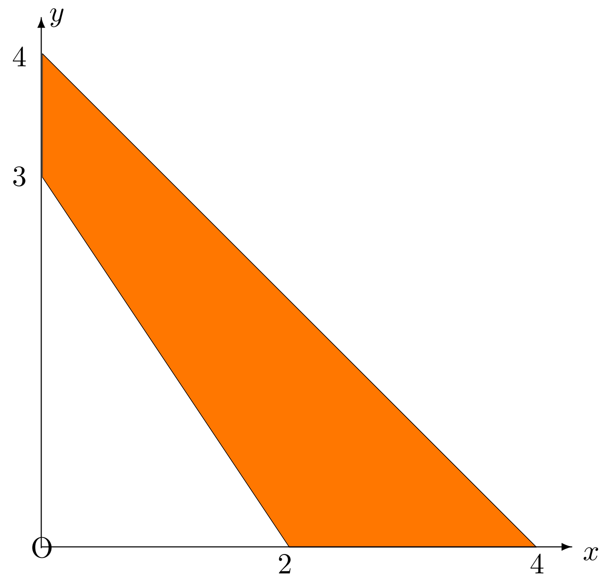
(d) The region is as follows:



2. The feasible region is as follows:



3. The feasible region is as follows:



The maximum value of $5x + 7y$ occurs at the point $(0, 4)$ and is equal to 28.

4. Subject to the simultaneous inequalities

$$x \geq 0,\ y \geq 0,\ 2x + y \geq 10,\ x + y \geq 7,\ 2x + 5y \geq 20 \text{ and } 4x + 5y \geq 45,$$

the function $2x + 5y$ has minimum value 20 at any point on the line $2x + 5y = 20$.

5. Subject to the simultaneous inequalities

$$x \geq 0,\ y \geq 0,\ x + 2y \leq 0 \text{ and } 3x + 2y \leq 80,$$

the objective function $P \equiv 8x + 10y$ has maximum value 260 at the point $(20, 10)$.