"JUST THE MATHS"

UNIT NUMBER

2.3

SERIES 3
(Elementary convergence and divergence)

by

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UNIT 2.3 - SERIES 3 - ELEMENTARY CONVERGENCE AND DIVERGENCE

Introduction

In the examination of geometric series in Unit 2.1 and of binomial series in Unit 2.2, the idea was introduced of series which have a first term but no last term; that is, there are an infinite number of terms.

The general format of an infinite series may be specified by either

$$u_1 + u_2 + u_3 + \ldots = \sum_{r=1}^{\infty} u_r$$

or

$$u_0 + u_1 + u_2 + \ldots = \sum_{r=0}^{\infty} u_r.$$

In the first of these two forms, u_r is the r-th term while, in the second, u_r is the $(r+1)$-th term.

ILLUSTRATIONS

1.

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{r=1}^{\infty} \frac{1}{r} = \sum_{r=0}^{\infty} \frac{1}{r+1}.$$

2.

$$2 + 4 + 6 + 8 + \ldots = \sum_{r=1}^{\infty} 2r = \sum_{r=0}^{\infty} 2(r+1).$$

3.

$$1 + 3 + 5 + 7 + \ldots = \sum_{r=1}^{\infty} (2r-1) = \sum_{r=0}^{\infty} (2r+1).$$

2.3.1 THE DEFINITIONS OF CONVERGENCE AND DIVERGENCE

It has already been shown in Unit 2.1 (for geometric series) that an infinite series may have a **"sum to infinity"** even though it is not possible to reach the end of the series.

For example, the infinite geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ =\sum_{r=1}^{\infty}\frac{1}{2^r}$$

is such that the sum, S_n, of the first n terms is given by

$$S_n=\frac{\frac{1}{2}(1-\frac{1}{2}^n)}{1-\frac{1}{2}}=1-\frac{1}{2^n}.$$

As n becomes larger and larger, S_n approaches ever closer to the fixed value, 1.

We say that the **"limiting value"** of S_n as n **"tends to infinity"** is 1; and we write

$$\lim_{n\to\infty} S_n=1.$$

Since this limiting value is a **finite** number, we say that the series **"converges"** to 1.

DEFINITION (A)
For the infinite series

$$\sum_{r=1}^{\infty} u_r,$$

the expression

$$u_1+u_2+u_3+\ldots\ +u_n$$

is called its **"n-th partial sum"**.

DEFINITION (B)
If the n-th partial sum of an infinite series tends to a finite limit as n tends to infinity, the series is said to **"converge"**. In **all** other cases, the series is said to **"diverge"**.

ILLUSTRATIONS

1.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ =\sum_{r=1}^{\infty}\ \frac{1}{2^r}\quad \text{converges.}$$

2.

$$1+2+3+4+\ldots.=\sum_{r=1}^{\infty} r\quad \text{diverges.}$$

3.

$$1-1+1-1+\ldots.=\sum_{r=1}^{\infty}\ (-1)^{n-1}\quad \text{diverges.}$$

Notes:

(i) The third illustration above shows that a series which diverges does not necessarily diverge to infinity.

(ii) Whether a series converges or diverges depends less on the starting terms than it does on the later terms. For example

$$7-15+2+39+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ ..$$

converges to $7-15+2+39+1=33+1=34$.

(iii) It will be seen, in the next section, that it is sometimes possible to test an infinite series for convergence or divergence without having to try and determine its sum to infinity.

2.3.2 TESTS FOR CONVERGENCE AND DIVERGENCE

In this section, the emphasis will be on the **use** of certain standard tests, rather than on their rigorous formal **proofs**. Only **outline** proofs will be suggested.

To begin with, we shall consider series of **positive** terms only.

TEST 1 - The r-th Term Test

An infinite series,

$$\sum_{r=1}^{\infty} u_r,$$

cannot converge unless its terms ultimately tend to zero; that is,

$$\lim_{r\to\infty} u_r = 0.$$

Outline Proof:

The series will converge only if the r-th partial sums, S_r, tend to a finite limit, L (say), as r tends to infinity; hence, if we observe that $u_r = S_r - S_{r-1}$, then u_r must tend to zero as r tends to infinity since S_r and S_{r-1} each tend to L.

ILLUSTRATIONS

1. The convergent series

$$\sum_{r=1}^{\infty} \frac{1}{2^r},$$

discussed earlier, is such that

$$\lim_{r\to\infty} \frac{1}{2^r} = 0.$$

2. The divergent series

$$\sum_{r=1}^{\infty} r,$$

discussed earlier, is such that

$$\lim_{r\to\infty} r \neq 0.$$

3. The series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is such that

$$\lim_{r\to\infty} \frac{1}{r} = 0,$$

but it will be shown later that this series is **divergent**.

That is, the converse of the r-th Term Test is not true. It does not imply that a series is convergent when its terms **do** tend to zero; merely that it is divergent when its terms **do not** tend to zero.

TEST 2 - The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\sum_{r=1}^{\infty} v_r$$

is a second series which is known to **converge**.

Then the first series converges provided that $u_r \leq v_r$.

Similarly, if

$$\sum_{r=1}^{\infty} w_r$$

is a series which is known to **diverge**, then the first series diverges provided that $u_r \geq w_r$.

Note:
It may be necessary to ignore the first few values of r.

Outline Proof:

Suppose we think of u_r and v_r as the heights of two sets of rectangles, all with a common base-length of one unit.

If the series

$$\sum_{r=1}^{\infty} v_r$$

is **convergent** it represents a **finite** total area of an infinite number of rectangles.

The series

$$\sum_{r=1}^{\infty} u_r$$

represents a **smaller** area and, hence, is also finite.

A similar argument holds when

$$\sum_{r=1}^{\infty} w_r$$

is a **divergent** series and $u_r \geq w_r$.

A divergent series of **positive** terms can diverge only to $+\infty$ so that the set of rectangles determined by u_r generates an area that is greater than an area which is already infinite.

EXAMPLES

1. Show that the series

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \ldots$$

   is divergent.

   Solution

   The given series may be written as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ \ldots,$$

a series whose terms are all greater than (or, for the second term, equal to) $\frac{1}{2}$. But the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ \ldots$$

is a divergent series and, hence, the series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent.

2. Given that

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is a convergent series, show that

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

is also a convergent series.

Solution

First, we observe that, for $r = 1, 2, 3, 4, \ldots,$

$$\frac{1}{r(r+1)} < \frac{1}{r.r} = \frac{1}{r^2}.$$

Hence, the terms of the series

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

are smaller in value than those of a known convergent series. It therefore converges also.

Note:
It may be shown that the series

$$\sum_{r=1}^{\infty} \frac{1}{r^p}$$

is convergent whenever $p > 1$ and divergent whenever $p \leq 1$. This result provides a useful standard tool to use with the Comparison Test.

TEST 3 - D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = L;$$

Then the series converges if $L < 1$ and diverges if $L > 1$.

There is no conclusion if $L = 1$.

Outline Proof:

(i) If $L > 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be greater than 1 and so $u_{r+1} > u_r$ for a large enough value of r.

Hence, the terms cannot ultimately be decreasing; so Test 1 shows that the series diverges.

(ii) If $L < 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be less than 1 and so $u_{r+1} < u_r$ for a large enough value or r.

We will consider that this first occurs when $r = s$; and, from this value onwards, the terms steadily decrease in value.

Furthermore, we can certainly find a positive number, h, between L and 1 such that

$$\frac{u_{s+1}}{u_s} < h, \frac{u_{s+2}}{u_{s+1}} < h, \frac{u_{s+3}}{u_{s+2}} < h, \ \ldots$$

That is,

$$u_{s+1} < hu_s, \ u_{s+2} < hu_{s+1}, \ u_{s+3} < hu_{s+2}, \ \ldots,$$

which gives

$$u_{s+1} < hu_s, \ u_{s+2} < h^2 u_s, \ u_{s+3} < h^3 u_s, \ \ldots$$

But, since $L < h < 1$,

$$hu_s + h^2 u_s + h^3 u_s + \ \ldots$$

is a convergent geometric series; therefore, by the Comparison Test,

$$u_{s+1} + u_{s+2} + u_{s+3} + \ \ldots = \sum_{r=1}^{\infty} \ u_{s+r} \ \text{ converges,}$$

implying that the original series converges also.

(iii) If $L = 1$, there will be no conclusion since we have already encountered examples of both a convergent series **and** a divergent series which give $L = 1$.

In particular,

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent and gives

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} \frac{r}{r+1} = \lim_{r \to \infty} \frac{1}{1 + \frac{1}{r}} = 1.$$

Also,

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is convergent and gives

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r^2}{(r+1)^2} = \lim_{r\to\infty} \left(\frac{r}{r+1}\right)^2 = \lim_{r\to\infty} \left(\frac{1}{1+\frac{1}{r}}\right)^2 = 1.$$

Note:
A convenient way to calculate the limit as r tends to infinity of any ratio of two polynomials in r is first to divide the numerator and the denominator by the highest power of r.

For example,

$$\lim_{r\to\infty} \frac{3r^3+1}{2r^3+1} = \lim_{r\to\infty} \frac{3+\frac{1}{r^3}}{2+\frac{1}{r^3}} = \frac{3}{2}.$$

ILLUSTRATIONS

1. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} \frac{r}{2^r},$$

$$\frac{u_{r+1}}{u_r} = \frac{r+1}{2^{r+1}}.\frac{2^r}{r} = \frac{r+1}{2r}.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r+1}{2r} = \lim_{r\to\infty} \frac{1+\frac{1}{r}}{2} = \frac{1}{2}.$$

The limiting value is less than 1 so that the series converges.

2. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} 2^r,$$

$$\frac{u_{r+1}}{u_r} = \frac{2^{r+1}}{2^r} = 2.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} 2 = 2.$$

The limiting value is greater than 1 so that the series diverges.

2.3.3 EXERCISES

1. Use the "r-th Term Test" to show that the following series are divergent:

   (a)

$$\sum_{r=1}^{\infty} \frac{r}{r+2};$$

   (b)

$$\sum_{r=1}^{\infty} \frac{1+2r^2}{1+r^2}.$$

2. Use the "Comparison Test" to determine whether the following series are convergent or divergent:

   (a)

$$\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)};$$

   (b)

$$\sum_{r=1}^{\infty} \frac{r}{\sqrt{r^6+1}};$$

(c)

$$\sum_{r=1}^{\infty} \frac{r}{r^2+1}.$$

3. Use D'Alembert's Ratio Test to determine whether the following series are convergent or divergent:

   (a)

$$\sum_{r=1}^{\infty} \frac{2^r}{r^2};$$

   (b)

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)!};$$

   (c)

$$\sum_{r=1}^{\infty} \frac{r+1}{r!}.$$

4. Obtain an expression for the r-th term of the following infinite series and, hence, investigate them for convergence or divergence:

   (a)

$$\frac{1}{1\times 2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} + \frac{1}{4\times 2^4} + \ . \ . \ ;$$

   (b)

$$1 + \frac{3}{2\times 4} + \frac{7}{4\times 9} + \frac{15}{8\times 16} + \frac{31}{16\times 25} + \ . \ . \ .;$$

   (c)

$$\frac{1}{\sqrt{3}-1} + \frac{1}{2-\sqrt{2}} + \frac{1}{\sqrt{5}-\sqrt{3}} + \frac{1}{\sqrt{6}-2} + \frac{1}{\sqrt{7}-\sqrt{5}} + \ . \ .$$

2.3.4 ANSWERS TO EXERCISES

1. (a)

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = 1 \neq 0;$$

   (b)

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = 2 \neq 0.$$

2. (a) Convergent;
   (b) Convergent;
   (c) Divergent.

3. (a) Divergent;
   (b) Convergent;
   (c) Convergent.

4. (a)

$$u_r = \frac{1}{r \times 2^r};$$

   The series is convergent by D'Alembert's Ratio Test;

   (b)

$$u_r = \frac{2^r - 1}{2^{r-1} r^2};$$

   The series is convergent by Comparison Test;

   (c)

$$u_r = \frac{1}{\sqrt{r+2} - \sqrt{r}};$$

   The series is divergent by r-th Term Test.

Note:
For further discussion of limiting values, see Unit 10.1