"JUST THE MATHS"

UNIT NUMBER

16.6

LAPLACE TRANSFORMS 6
(The Dirac unit impulse function)

by

A.J.Hobson

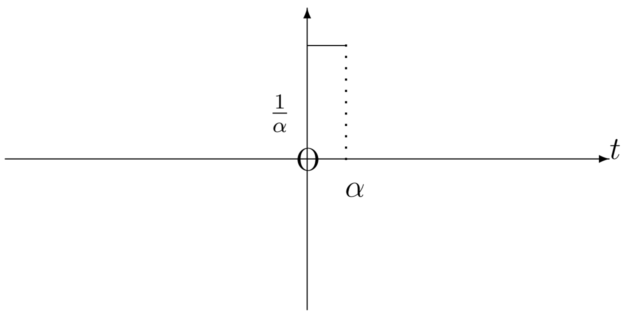
UNIT 16.6 - LAPLACE TRANSFORMS 6
THE DIRAC UNIT IMPULSE FUNCTION

16.6.1 THE DEFINITION OF THE DIRAC UNIT IMPULSE FUNCTION

A pulse of large magnitude, short duration and finite strength is called an **"impulse"**. In particular, a **"unit impulse"** is an impulse of strength 1.

ILLUSTRATION

Consider a pulse, of duration α, between $t = 0$ and $t = \alpha$, having magnitude, $\frac{1}{\alpha}$. The strength of the pulse is then 1.



From Unit 16.5, this pulse is given by

$$\frac{H(t) - H(t-\alpha)}{\alpha}.$$

If we now allow α to tend to zero, we obtain a unit impulse located at $t = 0$. This leads to the following definition:

DEFINITION 2

The **"Dirac unit impulse function"** , $\delta(t)$ is defined to be an impulse of unit strength located at $t = 0$. It is given by

$$\delta(t) = \lim_{\alpha \to 0} \frac{H(t) - H(t-\alpha)}{\alpha}.$$

Notes:

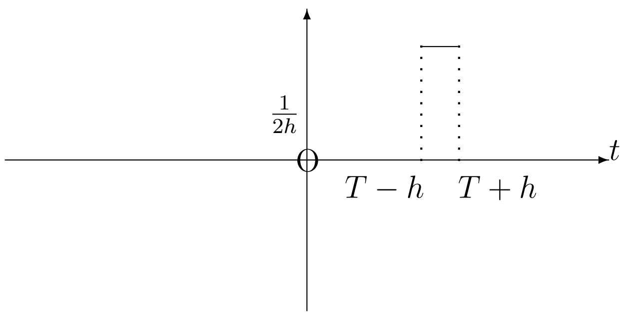
(i) An impulse of unit strength located at $t = T$ is represented by $\delta(t - T)$.

(ii) An alternative definition of the function $\delta(t - T)$ is as follows:

$$\delta(t - T) = \begin{cases} 0 & \text{for } t \neq T; \\ \infty & \text{for } t = T. \end{cases}$$

and

$$\lim_{h \to 0} \int_{T-h}^{T+h} \delta(t - T) \, \mathrm{d}t = 1.$$



THEOREM

$$\int_a^b f(t)\delta(t - T) \, \mathrm{d}t = f(T) \text{ if } a < T < b.$$

Proof:

Since $\delta(t - T)$ is equal to zero everywhere except at $t = T$, the left-hand side of the above formula reduces to

$$\lim_{h \to 0} \int_{T-h}^{T+h} f(t)\delta(t - T) \, \mathrm{d}t.$$

But, in the <u>small</u> interval from $T - h$ to $T + h$, $f(t)$ is approximately constant and equal to $f(T)$. Hence, the left-hand side may be written

$$f(T) \left[\lim_{h \to 0} \int_{T-h}^{T+h} \delta(T - T) \, \mathrm{d}t \right],$$

which reduces to $f(T)$, using note (ii) in the definition of the Dirac unit impulse function.

16.6.2 THE LAPLACE TRANSFORM OF THE DIRAC UNIT IMPULSE FUNCTION

RESULT

$$L[\delta(t-T)] = e^{-sT};$$

and, in particular,

$$L[\delta(t)] = 1.$$

Proof:

From the definition of a Laplace Transform,

$$L[\delta(t-T) = \int_0^\infty e^{-st}\delta(t-T) \, \mathrm{d}t.$$

But, from the Theorem discussed above, with $f(t) = e^{-st}$, we have

$$L[\delta(t-T)] = e^{-sT}.$$

EXAMPLES

1. Solve the differential equation,

   $$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \delta(t),$$

   given that $x = 0$ when $t = 0$.

   Solution

   Taking the Laplace Transform of the differential equation,

   $$3sX(s) + 4X(s) = 1.$$

   That is,

   $$X(s) = \frac{1}{3s+4} = \frac{1}{3}.\frac{1}{s+\frac{4}{3}}.$$

   Hence,

   $$x(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

2. Show that, for any function, $f(t)$,

$$\int_0^\infty f(t)\delta'(t-a)\ \mathrm{d}t = -f'(a).$$

Solution

Using Integration by Parts, the left-hand side of the formula may be written

$$[f(t)\delta(t-a)]_0^\infty - \int_0^\infty f'(t)\delta(t-a)\ \mathrm{d}t.$$

The first term of this reduces to zero, since $\delta(t-a)$ is equal to zero except when $t = a$. The required result follows from the Theorem discussed earlier, with $T = a$.

16.6.3 TRANSFER FUNCTIONS

In scientific applications, the solution of an ordinary differential equation having the form

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

is sometimes called the **"response of a system to the function** $f(t)$**"**.

The term **"system"** may, for example, refer to an oscillatory electrical circuit or a mechanical vibration.

It is also customary to refer to $f(t)$ as the **"input"** and $x(t)$ as the **"output"** of a system.

In the work which follows, we shall consider the special case in which $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$; that is, we shall assume zero initial conditions.

Impulse response function and transfer function

Consider, for the moment, the differential equation having the form,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + cu = \delta(t).$$

Here, we refer to the function, $u(t)$, as the **"impulse response function"** of the original system.

The Laplace Transform of its differential equation is given by

$$(as^2 + bs + c)U(s) = 1.$$

Hence,

$$U(s) = \frac{1}{as^2 + bs + c},$$

which is called the **"transfer function"** of the original system.

EXAMPLE

Determine the transfer function and impulse response function for the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = f(t),$$

assuming zero initial conditions.

Solution

To find $U(s)$ and $u(t)$, we have

$$3\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = \delta t,$$

so that

$$(3s + 4)U(s) = 1$$

and, hence, the transfer function is

$$U(s) = \frac{1}{3s + 4} = \frac{1}{3}.\frac{1}{s + \frac{4}{3}}.$$

Taking the inverse Laplace Transform of $U(s)$ gives the impulse response function,

$$u(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

System response for any input

Assuming zero initial conditions, the Laplace Transform of the differential equation

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + bx + cx = f(t)$$

is given by

$$(as^2 + bs + c)X(s) = F(s),$$

which means that

$$X(s) = \frac{F(s)}{as^2 + bs + c} = F(s).U(s).$$

In order to find the response of the system to the function $f(t)$, we need the inverse Laplace Transform of $F(s).U(s)$ which may possibly be found using partial fractions but may, if necessary, be found by using the Convolution Theorem referred to in Unit 16.1

The Convolution Theorem shows, in this case, that

$$L\left[\int_0^t f(T).u(t-T)\ \mathrm{d}T\right] = F(s).U(s);$$

in other words,

$$L^{-1}[F(s).U(s)] = \int_0^t f(T).u(t-T)\ \mathrm{d}T.$$

EXAMPLE

The impulse response of a system is known to be $u(t) = \frac{10e^{-t}}{3}$.

Determine the response, $x(t)$, of the system to an input of $f(t) \equiv \sin 3t$.

Solution

First, we note that

$$U(s) = \frac{10}{3(s+1)} \quad \text{and} \quad F(s) = \frac{3}{s^2+9}.$$

Hence,

$$X(s) = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} + \frac{-s+1}{s^2+9},$$

using partial fractions.

Thus

$$x(t) = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0.$$

Alternatively, using the Convolution Theorem,

$$x(t) = \int_0^t \sin 3T.\frac{10e^{-(t-T)}}{3}\ \mathrm{d}T;$$

but the integration here can be made simpler if we replace $\sin 3T$ by e^{j3T} and use the imaginary part, only, of the result.

Hence,

$$x(t) = I_m\left(\int_0^t \frac{10}{3}e^{-t}.e^{(1+j3)T}\ \mathrm{d}T\right)$$

$$= I_m\left(\frac{10}{3}\left[e^{-t}\frac{e^{(1+j3)T}}{1+j3}\right]_0^t\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{e^{-t}.e^{(1+j3)t} - e^{-t}}{1+j3}\right]\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{[(\cos 3t - e^{-t}) + j\sin 3t](1-j3)}{10}\right]\right)$$

$$= \frac{10}{3}\left[\frac{\sin 3t - 3\cos 3t + 3e^{-t}}{10}\right]$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0,$$

as before.

Note:
Clearly, in this example, the method using partial fractions is simpler.

16.6.4 STEADY-STATE RESPONSE TO A SINGLE FREQUENCY INPUT

In the differential equation,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

suppose that the quadratic denominator of the transfer function, $U(s)$, has negative real roots; that is, it gives rise to an impulse response, $u(t)$, involving negative powers of e and, hence, tending to zero as t tends to infinity.

Suppose also that $f(t)$ takes one of the forms, $\cos\omega t$ or $\sin\omega t$, which may be regarded, respectively, as the real and imaginary parts of the function, $e^{j\omega t}$.

It turns out that the response, $x(t)$, will consist of a **"transient"** part which tends to zero as t tends to infinity, together with a non-transient part forming the **"steady-state response"**.

We illustrate with an example:

EXAMPLE

Consider that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = e^{j7t},$$

where $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - 2) - 1 + 3(sX(s) - 2) + 2X(s) = \frac{1}{s - j7}.$$

That is,

$$(s^2 + 3s + 2)X(s) = 2s + 7 + \frac{1}{s - j7},$$

giving

$$X(s) = \frac{2s + 7}{s^2 + 3s + 2} + \frac{1}{(s - j7)(s^2 + 3s + 2)} = \frac{2s + 7}{(s + 2)(s + 1)} + \frac{1}{(s - j7)(s + 2)(s + 1)}.$$

Using the "cover-up" rule for partial fractions, we obtain

$$X(s) = \frac{5}{s + 1} - \frac{3}{s + 2} + \frac{1}{(-1 - j7)(s + 1)} + \frac{1}{(2 + j7)(s + 2)} + \frac{U(j7)}{(s - j7)},$$

where

$$U(s) \equiv \frac{1}{s^2 + 3s + 2}$$

is the transfer function.

Taking inverse Laplace Transforms,

$$x(t) = 5e^{-t} - 3e^{-2t} + \frac{1}{-1-j7}e^{-t} + \frac{1}{2+j7}e^{-2t} + U(j7)e^{j7t}.$$

The first four terms on the right-hand side tend to zero as t tends to infinity, so that the final term represents the steady state response; we need its real part if $f(t) \equiv \cos 7t$ and its imaginary part if $f(t) \equiv \sin 7t$.

Summary

The above example illustrates the result that the steady-state response, $s(t)$, of a system to an input of $e^{j\omega t}$ is given by

$$s(t) = U(j\omega)e^{j\omega t}.$$

16.6.5 EXERCISES

1. Evaluate
$$\int_0^\infty e^{-4t}\delta'(t-2)\ \mathrm{d}t.$$

2. In the following cases, solve the differential equation
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = f(t),$$
where $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$:

   (a)
$$f(t) \equiv \delta(t);$$

   (b)
$$f(t) \equiv \delta(t-2).$$

3. Determine the transfer function and impulse response function for the differential equation,
$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t),$$
assuming zero initial conditions.

4. The impulse response function of a system is known to be $u(t) = e^{3t}$. Determine the response, $x(t)$, of the system to an input of $f(t) \equiv 6\cos 3t$.

5. Determine the steady-state response to the system

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t)$$

in the cases when

(a)

$$f(t) \equiv e^{j2t};$$

(b)

$$f(t) \equiv 3\cos 2t.$$

16.6.6 ANSWERS TO EXERCISES

1.

$$4e^{-8}.$$

2. (a)

$$x = \sin 2t \quad t > 0;$$

(b)

$$x = \sin t + H(t-2)\sin(t-2) \quad t \neq 2.$$

3.

$$U(s) = \frac{1}{2s^2 - 3s + 1} \quad \text{and} \quad u(t) = [e^t - e^{\frac{1}{2}t}].$$

4.

$$\frac{1}{13}\left[18e^{3t} - 18\cos 2t + 12\sin 2t\right] \quad t > 0.$$

5. (a)

$$\frac{(1-j6)e^{j2t}}{37} \quad t > 0;$$

(b)

$$\frac{1}{37}(\cos 2t + 6\sin 2t) \quad t > 0.$$