"JUST THE MATHS"

UNIT NUMBER

16.9

Z-TRANSFORMS 2
(Inverse Z-Transforms)

by

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UNIT 16.9 - Z TRANSFORMS 2

INVERSE Z - TRANSFORMS

16.9.1 THE USE OF PARTIAL FRACTIONS

When solving linear difference equations by means of Z-Transforms, it is necessary to be able to determine a sequence, $\{u_n\}$, of numbers, whose Z-Transform is a known function, $F(z)$, of z. Such a sequence is called the **"inverse Z-Transform of $F(z)$"** and may be denoted by $\mathrm{Z}^{-1}[F(z)]$.

For simple difference equations, the function $F(z)$ turns out to be a rational function of z, and the method of partial fractions may be used to determine the corresponding inverse Z-Transform.

EXAMPLES

1. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{10z(z+5)}{(z-1)(z-2)(z+3)}.$$

Solution

Bearing in mind that

$$Z\{a^n\} = \frac{z}{z-a},$$

for any non-zero constant, a, we shall write

$$F(z) \equiv z.\left[\frac{10(z+5)}{(z-1)(z-2)(z+3)}\right],$$

which gives

$$F(z) \equiv z.\left[\frac{-15}{z-1} + \frac{14}{z-2} + \frac{1}{z+3}\right]$$

or

$$F(z) \equiv \frac{z}{z+3} + 14\frac{z}{z-2} - 15\frac{z}{z-1}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \{(-3)^n + 14(2)^n - 15\}.$$

2. Determine the Inverse Z-Transform of the function

$$F(z) \equiv \frac{1}{z-a}.$$

Solution

In this example, there is no factor, z, in the function $F(z)$ and we shall see that it is necessary to make use of the first shifting theorem.

First, we may write

$$F(z) \equiv \frac{1}{z}\left[\frac{z}{z-a}\right]$$

and, since the inverse Z-Transform of the expression inside the brackets is a^n, the first shifting theorem tells us that

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ a^{n-1} & \text{when } n > 0. \end{cases}$$

Note:

This may now be taken as a standard result.

3. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{4(2z+1)}{(z+1)(z-3)}.$$

Solution

Expressing $F(z)$ in partial fractions, we obtain

$$F(z) \equiv \frac{1}{z+1} + \frac{7}{(z-3)}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ (-1)^{n-1} + 7.(3)^{n-1} & \text{when } n > 0. \end{cases}$$

16.9.2 EXERCISES

1. Determine the inverse Z-Transforms of each of the following functions, $F(z)$:

   (a)
   $$F(z) \equiv \frac{z}{z-1};$$

   (b)
   $$F(z) \equiv \frac{z}{z+1};$$

   (c)
   $$F(z) \equiv \frac{2z}{2z-1};$$

   (d)
   $$F(z) \equiv \frac{z}{3z+1};$$

   (e)
   $$F(z) \equiv \frac{z}{(z-1)(z+2)};$$

   (f)
   $$F(z) \equiv \frac{z}{(2z+1)(z-3)};$$

   (g)
   $$F(z) \equiv \frac{z^2}{(2z+1)(z-1)}.$$

2. Determine the inverse Z-Transform of each of the following functions, $F(z)$, and list the first five terms of the sequence obtained:

   (a)
   $$F(z) \equiv \frac{1}{z-1};$$

   (b)
   $$F(z) \equiv \frac{z+2}{z+1};$$

(c)

$$F(z) \equiv \frac{z-3}{(z-1)(z-2)};$$

(d)

$$F(z) \equiv \frac{2z^2-7z+7}{(z-1)^2(z-2)}.$$

16.9.3 ANSWERS TO EXERCISES

1. (a)

$$\mathrm{Z}^{-1}[F(z)] = \{1\}$$

(b)

$$\mathrm{Z}^{-1}[F(z)] = \{(-1)^n\}$$

(c)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\left(\frac{1}{2}\right)^n\right\};$$

(d)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}\left(-\frac{1}{3}\right)^n\right\};$$

(e)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}[1-(-2)^n]\right\};$$

(f)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{7}\left[(3)^n - \left(-\frac{1}{2}\right)^n\right]\right\};$$

(g)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3} + \frac{1}{6}\left(-\frac{1}{2}\right)^n\right\}.$$

2. (a)

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 1 & \text{when } n > 0; \end{cases}$$

The first five terms are 0,1,1,1,1

(b)

$$Z^{-1}[F(z)] = \begin{cases} 1 & \text{when } n = 0; \\ (-1)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 1,1,−1,1,−1

(c)

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 2 - (2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,1,0,−2,−6

(d)

$$Z^{-1}[F(z] = \begin{cases} 0 & \text{when } n = 0; \\ 3 - 2n + (2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,2,1,1,3