"JUST THE MATHS"

UNIT NUMBER

16.10

Z-TRANSFORMS 3
(Solution of linear difference equations)

by

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UNIT 16.10 - Z TRANSFORMS 3

THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS

Linear difference equations may be solved by constructing the Z-Transform of both sides of the equation. The method will be illustrated with linear difference equations of the first and second orders (with constant coefficients).

16.10.1 FIRST ORDER LINEAR DIFFERENCE QUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+1} - 2u_n = (3)^{-n},$$

given that $u_0 = 2/5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z - \frac{1}{3}},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{2}{5}.\frac{z}{z-2} + \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}$$

$$\equiv \frac{2}{5}.\frac{z}{z-2} + z.\left[\frac{-\frac{3}{5}}{z-\frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$\equiv \frac{z}{z-2} - \frac{3}{5}.\frac{z}{z-\frac{1}{3}}.$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{(2)^n - \frac{3}{5}(3)^{-n}\right\}.$$

2. Solve the linear difference equation,

$$u_{n+1} + u_n = f(n),$$

given that

$$f(n) \equiv \begin{cases} 1 & \text{when } n = 0; \\ 0 & \text{when } n > 0. \end{cases}$$

and $u_0 = 5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.5$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - 5z + Z\{u_n\} = 1,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{1}{z+1} + \frac{5z}{z+1}.$$

Hence,

$$\{u_n\} = \begin{cases} 5 & \text{when } n = 0; \\ (-1)^{n-1} + 5(-1)^n \equiv 4(-1)^n & \text{when } n > 0. \end{cases}$$

16.10.2 SECOND ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation

$$u_{n+2} = u_{n+1} + u_n,$$

given that $u_0 = 0$ and $u_1 = 1$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n - z.0 \equiv z.Z\{u_n\}$$

and

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z.1 \equiv z^2 Z\{u_n\} - z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - z = z.Z\{u_n\} + Z\{u_n\},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{z}{z^2 - z - 1},$$

which may be written

$$Z\{u_n\} = \frac{z}{(z-\alpha)(z-\beta)},$$

where, from the quadratic formula,

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

Using partial fractions,

$$Z\{u_n\} = \frac{1}{\alpha - \beta}\left[\frac{z}{z-\alpha} - \frac{z}{z-\beta}\right].$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{\frac{1}{\alpha - \beta}\left[(\alpha)^n - (\beta)^n\right]\right\}.$$

2. Solve the linear difference equation

$$u_{n+2} - 7u_{n+1} + 10u_n = 16n,$$

given that $u_0 = 6$ and $u_1 = 2$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

and

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7\left[z.Z\{u_n\} - 6z\right] + 10Z\{u_n\} = \frac{16z}{(z-1)^2},$$

which, on rearrangement, gives

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2};$$

and, hence,

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

Using partial fractions, we obtain

$$Z\{u_n\} = z.\left[\frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1}\right].$$

The solution of the difference equation is therefore

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

3. Solve the linear difference equation

$$u_{n+2} + 2u_n = 0$$

given that $u_0 = 1$ and $u_1 = \sqrt{2}$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z^2 - z\sqrt{2}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2 Z\{u_n\} - z^2 - z\sqrt{2} + 2Z\{u_n\} = 0,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{z^2 + z\sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{(z + j\sqrt{2})(z - j\sqrt{2})}.$$

Using partial fractions,

$$Z\{u_n\} = z\left[\frac{\sqrt{2}(1+j)}{j2\sqrt{2}(z - j\sqrt{2})} + \frac{\sqrt{2}(1-j)}{-j2\sqrt{2}(z + j\sqrt{2})}\right] \equiv z.\left[\frac{(1-j)}{2(z - j\sqrt{2})} + \frac{(1+j)}{2(z + j\sqrt{2})}\right],$$

so that

$$\{u_n\} \equiv \left\{\frac{1}{2}(1-j)(j\sqrt{2})^n + \frac{1}{2}(1+j)(-j\sqrt{2})^n\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[(1-j)(j)^n + (1+j)(-j)^n\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[\sqrt{2}e^{-j\frac{\pi}{4}}.e^{j\frac{n\pi}{2}} + \sqrt{2}e^{j\frac{\pi}{4}}.e^{-j\frac{n\pi}{2}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}\left[e^{j\frac{(2n-1)\pi}{4}} + e^{-j\frac{(2n-1)\pi}{4}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}.2\cos\frac{(2n-1)\pi}{4}\right\}$$

$$\equiv \left\{(\sqrt{2})^{n+1}\cos\frac{(2n-1)\pi}{4}\right\}.$$

16.10.3 EXERCISES

1. Solve the following first-order linear difference equations:

   (a)
   $$3u_{n+1} + 2u_n = (-1)^n,$$
   given that $u_0 = 0$;

   (b)
   $$u_{n+1} - 5u_n = 3(2)^n,$$
   given that $u_0 = 1$;

   (c)
   $$u_{n+1} + u_n = n,$$
   given that $u_0 = 1$;

   (d)
   $$u_{n+1} + 2u_n = f(n),$$
   where
   $$f(n) \equiv \begin{cases} 3 & \text{when } n = 0; \\ 0 & \text{when } n > 0; \end{cases}$$
   and $u_0 = 2$;

   (e)
   $$u_{n+1} - 3u_n = \sin\frac{n\pi}{2} + \frac{1}{2}\cos\frac{n\pi}{2},$$
   given that $u_0 = 0$.

2. Solve the following second-order linear difference equations:

   (a)
   $$u_{n+2} - 2u_{n+1} + u_n = 0,$$
   given that $u_0 = 0$ and $u_1 = 1$;

   (b)
   $$u_{n+2} - 4u_n = n,$$
   given that $u_0 = 0$ and $u_1 = 1$;

(c)

$$u_{n+2} - 8u_{n+1} - 9u_n = 24,$$

given that $u_0 = 2$ and $u_1 = 0$;

(d)

$$6u_{n+2} + 5u_{n+1} - u_n = 20,$$

given that $u_0 = 3$ and $u_1 = 8$;

(e)

$$u_{n+2} + 2u_{n+1} - 15u_n = 32\cos n\pi,$$

given that $u_0 = 0$ and $u_1 = 0$;

(f)

$$u_{n+2} - 3u_{n+1} + 3u_n = 5,$$

given that $u_0 = 5$ and $u_1 = 8$.

16.10.4 ANSWERS TO EXERCISES

1. (a)

$$\{u_n\} \equiv \left\{\left(-\frac{2}{3}\right)^n - (-1)^n\right\};$$

(b)

$$\{u_n\} \equiv \{2(5)^n - (2)^n\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}n - \frac{1}{4} + \frac{5}{4}(-1)^n\right\};$$

(d)

$$\{u_n\} \equiv 2(-2)^n + 3(-2)^{n-1} \quad \text{when} \quad n > 0;$$

(e)

$$\{u_n\} \equiv \left\{\frac{1}{4}\left[(3^n - \sqrt{2}\cos\frac{(2n-1)\pi}{4}\right]\right\}.$$

2. (a)

$$\{u_n\} \equiv \{n\};$$

(b)

$$\{u_n\} \equiv \left\{\frac{1}{2}(2)^n - \frac{1}{3}n - \frac{5}{18}(-2)^n - \frac{2}{9}\right\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}(9)^n + 3(-1)^n - \frac{3}{2}\right\};$$

(d)

$$\{u_n\} \equiv \left\{2 + (6)^{1-n} - 5(-1)^n\right\};$$

(e)

$$\{u_n\} \equiv \left\{2(-1)^{n+1} + (3)^n + (-5)^n\right\};$$

(f)

$$\{u_n\} \equiv \left\{5 + (2\sqrt{3})^{n+1} \cos\frac{(n-3)\pi}{6}\right\}.$$