

“JUST THE MATHS”

UNIT NUMBER

15.10

**ORDINARY
DIFFERENTIAL EQUATIONS 10
(Simultaneous equations (C))**

by

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15.10.1 Matrix methods for non-homogeneous systems
15.10.2 Exercises
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which may be interpreted as

$$\frac{d\mathbf{X}}{dt} = \mathbf{M}\mathbf{X} + \mathbf{N}g(t) \quad \text{where } \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \quad \text{and } \mathbf{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding “homogeneous” system

$$\frac{d\mathbf{X}}{dt} = \mathbf{M}\mathbf{X},$$

for which the characteristic equation is

$$\begin{vmatrix} 0 - \lambda & 1 \\ -4 & -5 - \lambda \end{vmatrix} = 0,$$

and gives

$$\lambda(5 + \lambda) + 4 = 0 \quad \text{or} \quad \lambda^2 + 5\lambda + 4 = 0 \quad \text{or} \quad (\lambda + 1)(\lambda + 4) = 0.$$

(iii) The eigenvectors of \mathbf{M} are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5 + \lambda)k_2 &= 0. \end{aligned}$$

Hence, in the case when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0, \end{aligned}$$

and these are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned}4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0\end{aligned}$$

which are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{dX}{dt} = MX + Ng(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form. The three cases in this example are as follows:

(a) $g(t) \equiv t$

$$\text{Trial solution } X = P + Qt,$$

where P and Q are constant matrices of order 2×1 .

We require that

$$Q = M(P + Qt) + Nt,$$

whereupon, equating the matrix coefficients of t and the constant matrices,

$$MQ + N = \mathbf{0} \quad \text{and} \quad Q = MP,$$

giving

$$Q = -M^{-1}N \quad \text{and} \quad P = M^{-1}Q.$$

Thus, using

$$M^{-1} = \frac{1}{4} \begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$Q = -\frac{1}{4} \begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$P = \frac{1}{4} \begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

Trial solution $X = Pe^{2t}$

We require that

$$2Pe^{2t} = MPe^{2t} + Ne^{2t}.$$

That is,

$$2P = MP + N.$$

The matrix, P, may now be determined from the formula

$$(2I - M)P = N;$$

or, in more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix} \cdot P = N.$$

Hence,

$$P = \frac{1}{18} \begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{18} \begin{bmatrix} 7 \\ -4 \end{bmatrix} e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution } X = P \sin t + Q \cos t.$$

We require that

$$P \cos t - Q \sin t = M(P \sin t + Q \cos t) + N \sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$P = MQ \quad \text{and} \quad -Q = MP + N,$$

which means that

$$-Q = M^2Q + N \quad \text{or} \quad (M^2 + I)Q = -N.$$

Thus,

$$Q = -(M^2 + I)^{-1}N,$$

where

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

and, hence,

$$Q = -\frac{1}{34} \begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Also,

$$P = MQ = \frac{1}{34} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin t + \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix} \cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1 .

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t},$$

whereupon, equating the matrix coefficients of te^{-t} and e^{-t} , we obtain

$$-Q = MQ \quad \text{and} \quad Q - P = MP + N.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding to the eigenvalue -1 and so, from earlier work,

$$Q = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any constant k .

Also,

$$(M + I)P = Q - N;$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Using $p_1 + p_2 = k$ and $-4p_1 - 4p_2 = -k - 1$, we deduce that $k = \frac{1}{3}$ and that the matrix P is given by

$$P = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + l \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any number, l .

Taking $l = 0$ for simplicity, a particular integral is therefore

$$X = \frac{1}{3} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t \right\} e^{-t}.$$

and the general solution is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{3} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t \right\} e^{-t}.$$

Note:

In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{dX}{dt} = MX + N_1 f(t) + N_2 g(t),$$

where

$$N_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$X = P + Qt + Re^{2t},$$

where P , Q and R are matrices of order 2×1 .

15.10.2 EXERCISES

1. Determine the general solutions of the following systems of simultaneous differential equations:

(a)

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 + 3x_2 + 5t, \\ \frac{dx_2}{dt} &= 3x_1 + x_2 + e^{3t}. \end{aligned}$$

(b)

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 + 2x_2 + t^2, \\ \frac{dx_2}{dt} &= 4x_1 + x_2 + e^{-2t}.\end{aligned}$$

2. Determine the complete solutions of the following systems of differential equations, subject to the conditions given:

(a)

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + 9x_2 + 3, \\ \frac{dx_2}{dt} &= 11x_1 + x_2 + e^{10t},\end{aligned}$$

given that $x_1 = \frac{1}{225}$ and $x_2 = -\frac{1}{100}$ when $t = 0$.

(b)

$$\begin{aligned}\frac{dx_1}{dt} &= 5x_1 + 2x_2 + 2t^2 + t, \\ \frac{dx_2}{dt} &= -2x_1 + x_2,\end{aligned}$$

given that $x_1 = \frac{32}{27}$ and $x_2 = -\frac{12}{27}$ when $t = 0$.

(c)

$$\begin{aligned}\frac{dx_1}{dt} &= 8x_1 + x_2 + \sin t, \\ \frac{dx_2}{dt} &= -5x_1 + 6x_2 + \cos t,\end{aligned}$$

given that $x_1 = 0$ and $x_2 = 0$ when $t = 0$.

15.10.3 ANSWERS TO EXERCISES

1. (a)

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + B \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + \frac{1}{32} \begin{bmatrix} -25 \\ 15 \end{bmatrix} + \frac{1}{8} \begin{bmatrix} 5 \\ -15 \end{bmatrix} t - \frac{1}{5} \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t};$$

(b)

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + B \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + \frac{2}{125} \begin{bmatrix} 41 \\ -84 \end{bmatrix} + \frac{2}{25} \begin{bmatrix} -9 \\ 16 \end{bmatrix} t + \frac{1}{5} \begin{bmatrix} 1 \\ -4 \end{bmatrix} t^2 + \frac{1}{7} \begin{bmatrix} 2 \\ -5 \end{bmatrix} e^{-2t}.$$

2. (a)

$$-\frac{7}{45} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-10t} + \frac{13}{900} \begin{bmatrix} 9 \\ 11 \end{bmatrix} e^{10t} + \frac{3}{100} \begin{bmatrix} 1 \\ -11 \end{bmatrix} + \frac{1}{180} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{10t} + \frac{1}{20} \begin{bmatrix} 9 \\ 11 \end{bmatrix} t e^{10t};$$

(b)

$$\left\{ (2t + 1) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} e^{3t} + \frac{1}{27} \begin{bmatrix} 5 \\ -12 \end{bmatrix} + \frac{1}{27} \begin{bmatrix} 1 \\ -22 \end{bmatrix} t - \frac{2}{9} \begin{bmatrix} 1 \\ 2 \end{bmatrix} t^2;$$

(c)

$$\frac{1}{145} \left\{ e^{7t} \left(\begin{bmatrix} -1 \\ 25 \end{bmatrix} \cos 2t + \begin{bmatrix} -12 \\ -10 \end{bmatrix} \sin 2t \right) + \begin{bmatrix} -17 \\ -10 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ -25 \end{bmatrix} \cos t \right\}.$$