

“JUST THE MATHS”

UNIT NUMBER

13.9

**INTEGRATION APPLICATIONS 9
(First moments of a surface of revolution)**

by

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UNIT 13.9 - INTEGRATION APPLICATIONS 9

FIRST MOMENTS OF A SURFACE OF REVOLUTION

13.9.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy -plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x -axis, the “**first moment**” about a plane through the origin, perpendicular to the x -axis, is given by

$$\lim_{\delta s \rightarrow 0} \sum_C 2\pi xy \delta s,$$

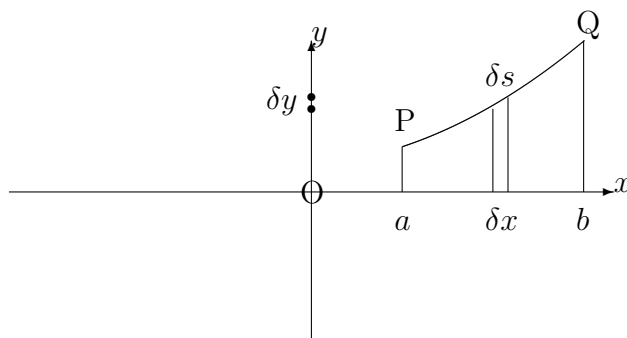
where x is the perpendicular distance, from the plane of moments, of the thin band, with surface area $2\pi y \delta s$, so generated.

13.9.2 INTEGRATION FORMULAE FOR FIRST MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs , by using neighbouring

points along the arc, separated by typical distances of δx (parallel to the x -axis) and δy (parallel to the y -axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x;$$

so that, for the surface of revolution of the arc about the x -axis, the first moment becomes

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} 2\pi xy \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x = \int_a^b 2\pi xy \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Hence,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\frac{dx}{dt}},$$

provided that $\frac{dx}{dt}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi xy \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{t_1}^{t_2} 2\pi xy \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dt} dt,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the first moment about the plane through the origin, perpendicular to the x -axis is given by

$$\text{First Moment} = \pm \int_{t_1}^{t_2} 2\pi xy \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

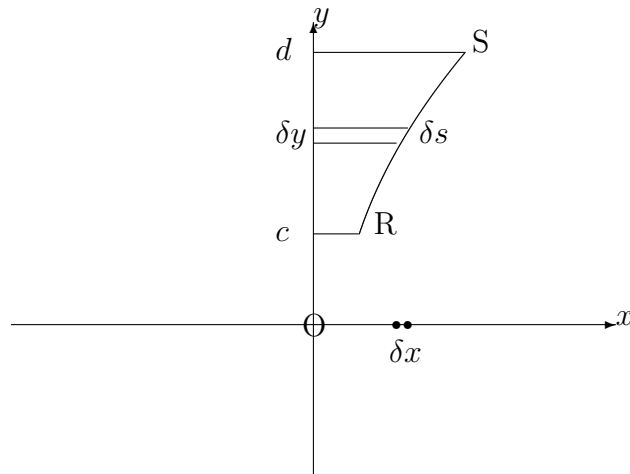
according as $\frac{dx}{dt}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the first moment about a plane through the origin, perpendicular to the y -axis is given by

$$\int_c^d 2\pi yx \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$



Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the first moment about a plane through the origin, perpendicular to the y -axis, is given by

$$\text{First moment} = \pm \int_{t_1}^{t_2} 2\pi y x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as $\frac{dy}{dt}$ is positive or negative.

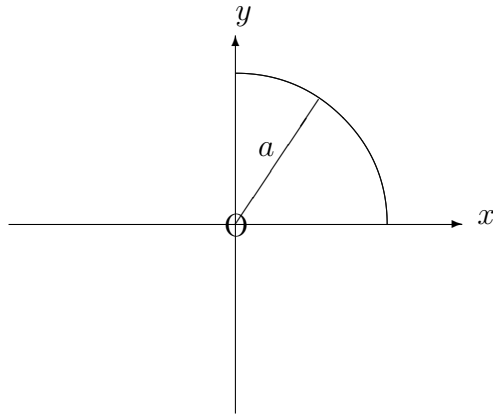
EXAMPLES

1. Determine the first moment about a plane through the origin, perpendicular to the x -axis, for the hemispherical surface of revolution (about the x -axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y \frac{dy}{dx} = 0$$

and hence, $\frac{dy}{dx} = -\frac{x}{y}$.

The first moment about the specified plane is therefore given by

$$\int_0^a 2\pi xy \sqrt{1 + \frac{x^2}{y^2}} dx = \int_0^a 2\pi xy \sqrt{\frac{x^2 + y^2}{y^2}} dx.$$

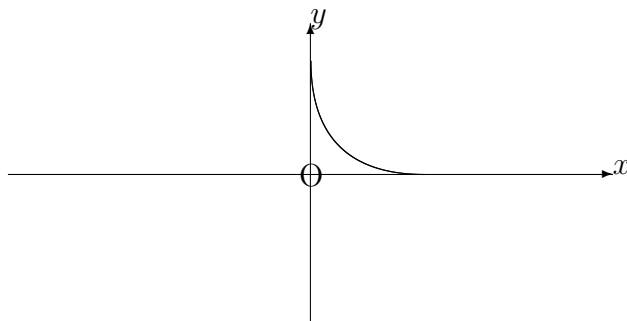
But $x^2 + y^2 = a^2$, and so the first moment becomes

$$\int_0^a 2\pi ax dx = [\pi ax^2]_0^a = \pi a^3.$$

- Determine the first moments about planes through the origin, (a) perpendicular to the x -axis and (b) perpendicular to the y -axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{dx}{d\theta} = -3a\cos^2\theta \sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\sin^2\theta \cos\theta.$$

Hence, the first moment about the x -axis is given by

$$- \int_{\frac{\pi}{2}}^0 2\pi xy \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta} \, d\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^2 \cos^3\theta \sin^3\theta \cdot 3a \cos\theta \sin\theta \, d\theta = \int_0^{\frac{\pi}{2}} 6\pi a^3 \cos^4\theta \sin^4\theta \, d\theta.$$

Using $2\sin\theta \cos\theta \equiv \sin 2\theta$, the integral reduces to

$$\frac{3\pi a^3}{8} \int_0^{\frac{\pi}{2}} \sin^4 2\theta \, d\theta,$$

which, by the methods of Unit 12.7, gives

$$\frac{3\pi a^3}{32} \int_0^{\frac{\pi}{2}} \left(1 - 2\cos 4\theta + \frac{1 + \cos 8\theta}{2} \right) d\theta = \frac{3\pi a^3}{32} \left[\frac{3\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16} \right]_0^{\frac{\pi}{2}} = \frac{9\pi a^3}{128}.$$

By symmetry, or by direct integration, the first moment about a plane through the origin, perpendicular to the y -axis is also $\frac{9\pi a^3}{128}$.

13.9.3 THE CENTROID OF A SURFACE OF REVOLUTION

Having calculated the first moment of a surface of revolution about a plane through the origin, perpendicular to the x -axis, it is possible to determine a point, $(\bar{x}, 0)$, on the x -axis with the property that the first moment is given by $S\bar{x}$, where S is the total surface area.

The point is called the “**centroid**” or the “**geometric centre**” of the surface of revolution and, for the surface of revolution of the arc of the curve whose equation is $y = f(x)$, between $x = a$ and $x = b$, the value of \bar{x} is given by

$$\bar{x} = \frac{\int_a^b 2\pi xy \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}{\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx} = \frac{\int_a^b xy \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}{\int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}.$$

Note:

The centroid effectively tries to concentrate the whole surface at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass of a thin sheet, for example, with uniform density.

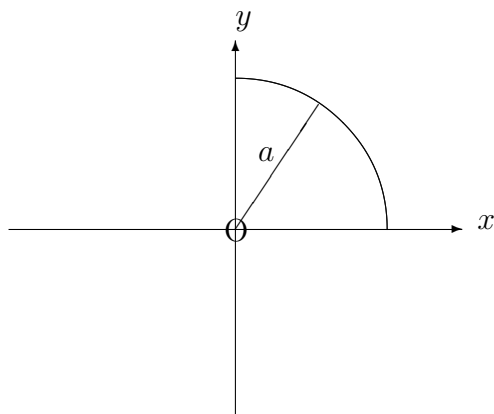
EXAMPLES

1. Determine the position of the centroid of the surface of revolution (about the x -axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 of Section 13.9.2, we know that the first moment of the surface about a plane through the origin, perpendicular to the the x -axis is equal to πa^3 .

Also, the total surface area is

$$\int_0^a 2\pi y \sqrt{1 + \frac{x^2}{y^2}} dx = \int_0^a 2\pi a dx = 2\pi a^2,$$

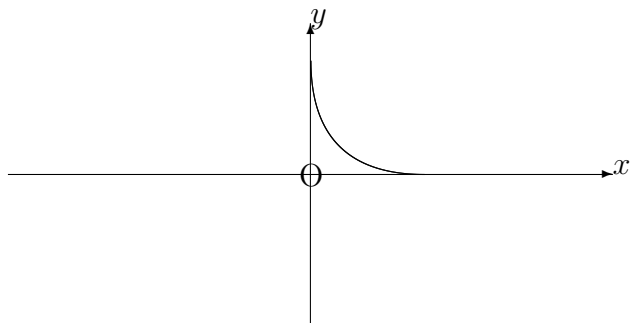
which implies that

$$\bar{x} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

2. Determine the position of the centroid of the surface of revolution (about the x -axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 of Section 13.9.2 that the first moment of the surface about a plane through the origin, perpendicular to the x -axis is equal to $\frac{9\pi a^3}{128}$.

Also, the total surface area is given by

$$- \int_{\frac{\pi}{2}}^0 2\pi a \sin^3 \theta \cdot 3a \cos \theta \sin \theta \, d\theta = \int_0^{\frac{\pi}{2}} 3a^2 \sin^4 \theta \cos \theta \, d\theta = 3\pi a^2 \left[\frac{\sin^5 \theta}{5} \right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$\bar{x} = \frac{15a}{128}.$$

13.9.4 EXERCISES

1. Determine the first moment, about a plane through the origin, perpendicular to the x -axis, of the surface of revolution (about the x -axis) of the straight-line segment joining the origin to the point $(3, 4)$.
2. Determine the first moment about a plane through the origin, perpendicular to the x -axis, of the surface of revolution (about the x -axis) of the arc of the curve whose equation is

$$y^2 = 4x,$$

lying between $x = 0$ and $x = 1$.

3. Determine the first moment about a plane through the origin, perpendicular to the y -axis, of the surface of revolution (about the y -axis) of the arc of the curve whose equation is

$$y^2 = 4(x - 1),$$

lying between $y = 2$ and $y = 4$.

4. Determine the first moment, about a plane through the origin, perpendicular to the y -axis, of the surface of revolution (about the y -axis) of the arc of the curve whose parametric equations are

$$x = 2 \cos t, \quad y = 3 \sin t,$$

joining the point $(2, 0)$ to the point $(0, 3)$.

5. Determine the position of the centroid of a hollow right-circular cone with height h .
6. For the curve whose equation is

$$9y^2 = x(3 - x)^2,$$

show that

$$\frac{dy}{dx} = \frac{1 - x}{2\sqrt{x}}.$$

Hence, show that the centroid of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x -axis lies at the point $\left(\frac{5}{4}, 0\right)$.

13.9.5 ANSWERS TO EXERCISES

1.

$$40\pi.$$

2.

$$4\pi \left[\frac{12\sqrt{2}}{5} - \frac{4}{15} \right] \simeq 39.3$$

3.

$$\left[\frac{8\pi}{5} \left(1 + \frac{y^2}{4} \right)^{\frac{5}{2}} \right]_2^4 \simeq 41.98$$

4.

$$\left[-\frac{4\pi}{5} \left(4 + 5\cos^2 t \right)^{\frac{3}{2}} \right]_0^{\frac{\pi}{2}} \simeq 47.75$$

5. Along the central axis, at a distance of $\frac{2h}{3}$ from the vertex.

6.

$$\text{First moment} = \frac{15\pi}{4} \quad \text{Surface Area} = 3\pi.$$