

“JUST THE MATHS”

UNIT NUMBER

13.7

INTEGRATION APPLICATIONS 7
(First moments of an area)

by

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13.7.1 Introduction

13.7.2 First moment of an area about the y -axis

13.7.3 First moment of an area about the x -axis

13.7.4 The centroid of an area

13.7.5 Exercises

13.7.6 Answers to exercises

UNIT 13.7 - INTEGRATION APPLICATIONS 7

FIRST MOMENTS OF AN AREA

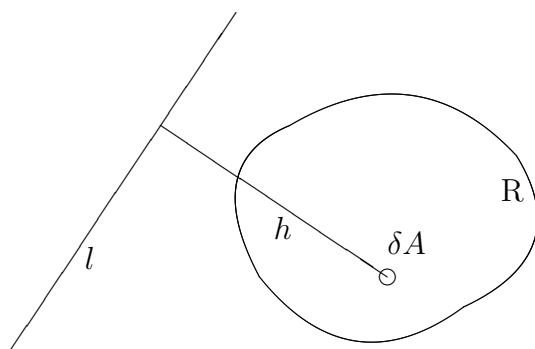
13.7.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy -plane of cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the “**first moment**” of R about a fixed line, l , in the plane of R is given by

$$\lim_{\delta A \rightarrow 0} \sum_R h \delta A,$$

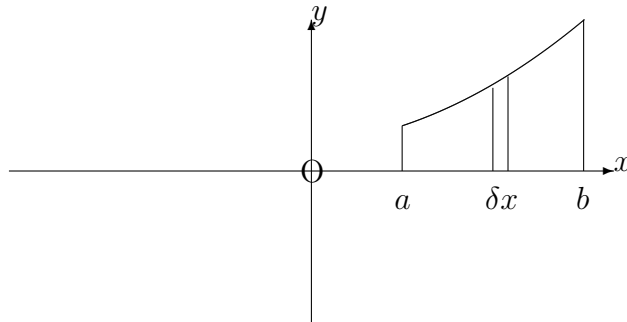
where h is the perpendicular distance, from l , of the element with area, δA .



13.7.2 FIRST MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy -plane, bounded by the x -axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may be divided up into small elements by using a network, consisting of neighbouring lines parallel to the y -axis and neighbouring lines parallel to the x -axis.

But all of the elements in a narrow 'strip' of width δx and height y (parallel to the y -axis) have the same perpendicular distance, x , from the y -axis.

Hence the first moment of this strip about the y -axis is x times the area of the strip; that is, $x(y\delta x)$, implying that the total first moment of the region about the y -axis is given by

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy \, dx.$$

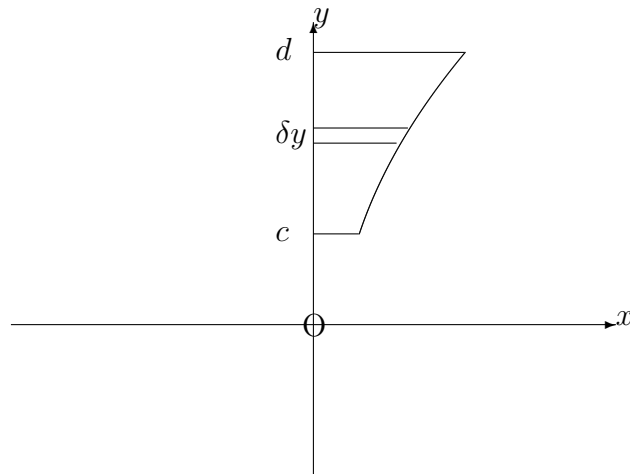
Note:

First moments about the x -axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y -axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment about the x -axis is given by

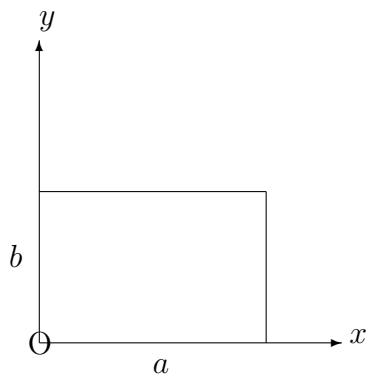
$$\int_c^d yx \, dy.$$



EXAMPLES

1. Determine the first moment of a rectangular region, with sides of lengths a and b about the side of length b .

Solution



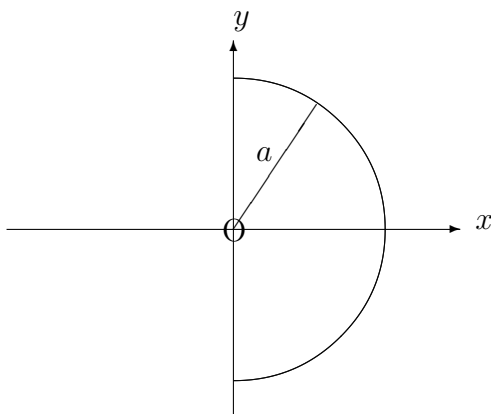
The first moment about the y -axis is given by

$$\int_0^a xb \, dx = \left[\frac{x^2b}{2} \right]_0^a = \frac{1}{2}a^2b.$$

2. Determine the first moment about the y -axis of the semi-circular region, bounded in the first and fourth quadrants by the y -axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the first moment about the y -axis is given by

$$2 \int_0^a x\sqrt{a^2 - x^2} \, dx = \left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}} \right]_0^a = \frac{2}{3}a^3.$$

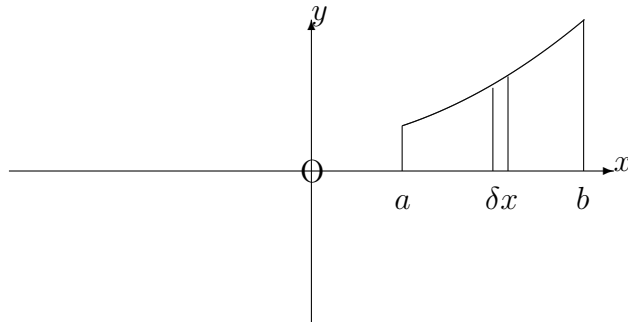
Note:

Although first moments about the x -axis will be discussed mainly in the next section of this Unit, we note that the symmetry of the above region shows that its first moment about the x -axis would be zero; this is because, for each $y(x\delta y)$, there will be a corresponding $-y(x\delta y)$ in calculating the first moments of the strips parallel to the x -axis.

13.7.3 FIRST MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the first moment of a rectangular region about one of its sides. This result may now be used to determine the first moment about the x -axis of a region enclosed in the first quadrant by the x -axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow strip, of width δx and height y , is regarded as approximately a rectangle, its first moment about the x -axis is $\frac{1}{2}y^2\delta x$. Hence, the first moment of the whole region about the x -axis is given by

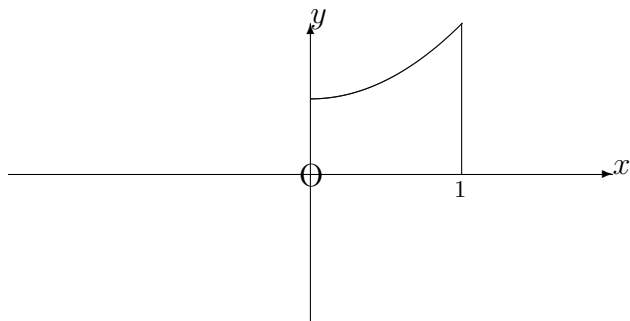
$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} \frac{1}{2}y^2\delta x = \int_a^b \frac{1}{2}y^2 dx.$$

EXAMPLES

1. Determine the first moment about the x -axis of the region, bounded in the first quadrant, by the x -axis, the y -axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution

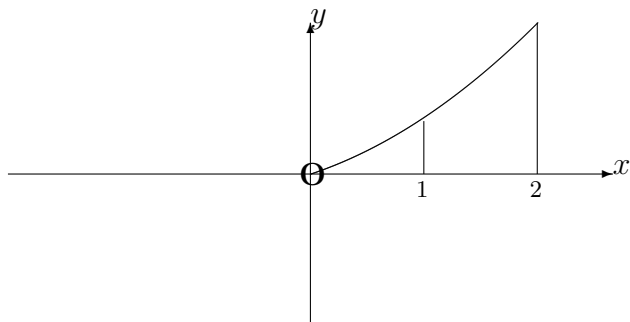


$$\text{First moment} = \int_0^1 \frac{1}{2}(x^2 + 1)^2 dx = \frac{1}{2} \int_0^1 (x^4 + 2x^2 + 1) dx = \frac{1}{2} \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_0^1 = \frac{28}{15}.$$

2. Determine the first moment about the x -axis of the region, bounded in the first quadrant, by the x -axis, the lines $x = 1$, $x = 2$ and the curve

$$y = xe^x.$$

Solution



$$\text{First moment} = \int_1^2 \frac{1}{2}x^2e^{2x} dx$$

$$\begin{aligned}
&= \frac{1}{2} \left(\left[x^2 \frac{e^{2x}}{2} \right]_1^2 - \int_1^2 x e^{2x} dx \right) \\
&= \frac{1}{2} \left(\left[x^2 \frac{e^{2x}}{2} \right]_1^2 - \left[x \frac{e^{2x}}{2} \right]_1^2 + \int_1^2 \frac{e^{2x}}{2} dx \right).
\end{aligned}$$

That is,

$$\frac{1}{2} \left[x^2 \frac{e^{2x}}{2} - x \frac{e^{2x}}{2} + \frac{e^{2x}}{4} \right]_1^2 = \frac{5e^4 - e^2}{8} \simeq 33.20$$

13.7.4 THE CENTROID OF AN AREA

Having calculated the first moments of a two dimensional region about both the x -axis and the y -axis, it is possible to determine a point, (\bar{x}, \bar{y}) , in the xy -plane with the property that

(a) The first moment about the y -axis is given by $A\bar{x}$, where A is the total area of the region;

and

(b) The first moment about the x -axis is given by $A\bar{y}$, where A is the total area of the region.

The point is called the “**centroid**” or the “**geometric centre**” of the region and, in the case of a region bounded, in the first quadrant, by the x -axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, its co-ordinates are given by

$$\bar{x} = \frac{\int_a^b xy dx}{\int_a^b y dx} \quad \text{and} \quad \bar{y} = \frac{\int_a^b \frac{1}{2}y^2 dx}{\int_a^b y dx}.$$

Notes:

(i) The first moment of an area, about an axis through its centroid will, by definition, be zero. In particular, if we take the y -axis to be parallel to the given axis, with x as the perpendicular distance from an element, δA , to the y -axis, the first moment about the given axis will be

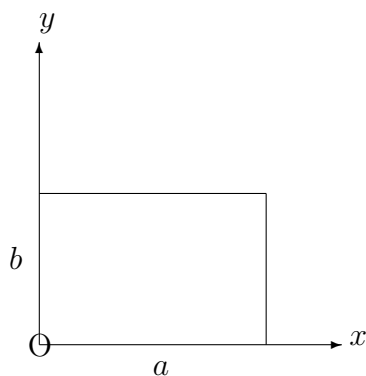
$$\sum_{\text{R}} (x - \bar{x})\delta A = \sum_{\text{R}} x\delta A - \bar{x} \sum_{\text{R}} \delta A = A\bar{x} - A\bar{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole area at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass for a thin plate with uniform density, whose shape is that of the region which we have been considering.

EXAMPLES

1. Determine the position of the centroid of a rectangular region with sides of lengths, a and b .

Solution



The area of the rectangle is ab and, from Example 1 in section 13.7.2, the first moments about the y -axis and the x -axis are $\frac{1}{2}a^2b$ and $\frac{1}{2}b^2a$, respectively.

Hence,

$$\bar{x} = \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$

and

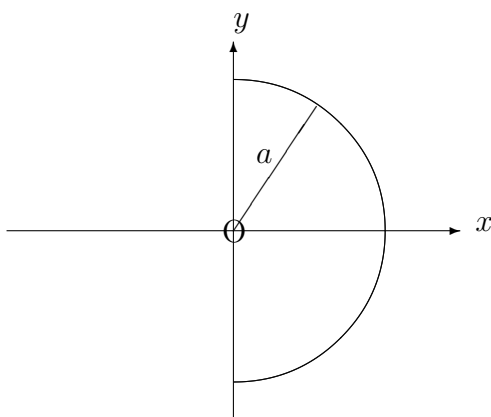
$$\bar{y} = \frac{\frac{1}{2}b^2a}{ab} = \frac{1}{2}b,$$

as we would expect for a rectangle.

2. Determine the position of the centroid of the semi-circular region bounded, in the first and fourth quadrants, by the y -axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



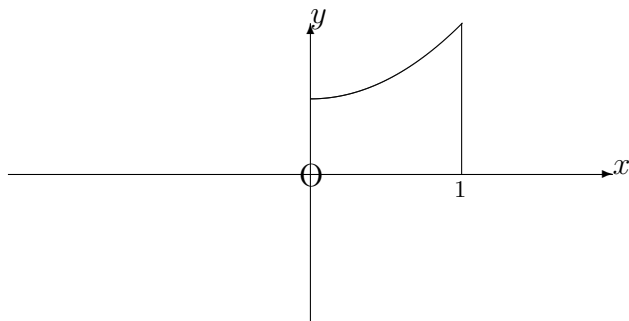
The area of the semi-circular region is $\frac{1}{2}\pi a^2$ and so, from Example 2, in section 13.7.2,

$$\bar{x} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \quad \text{and} \quad \bar{y} = 0.$$

3. Determine the position of the centroid of the region, bounded in the first quadrant, by the x -axis, the y -axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



The first moment about the y -axis is given by

$$\int_0^1 x(x^2 + 1) \, dx = \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^1 = \frac{3}{4}.$$

The area is given by

$$\int_0^1 (x^2 + 1) \, dx = \left[\frac{x^3}{3} + x \right]_0^1 = \frac{4}{3}.$$

Hence,

$$\bar{x} = \frac{3}{4} \div \frac{4}{3} = 1.$$

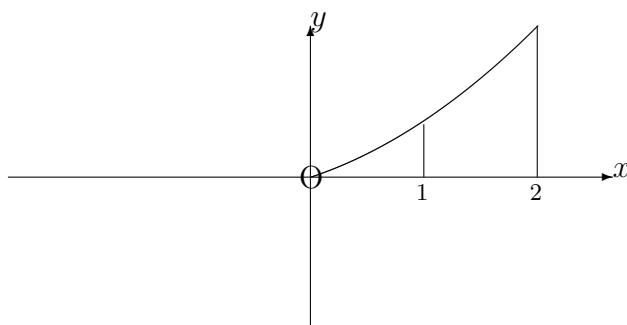
The first moment about the x -axis is $\frac{28}{15}$, from Example 1 in section 13.7.3; and, therefore,

$$\bar{y} = \frac{28}{15} \div \frac{4}{3} = \frac{7}{5}.$$

4. Determine the position of the centroid of the region bounded in the first quadrant by the x -axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



The first moment about the y -axis is given by

$$\int_1^2 x^2 e^x \, dx = [x^2 e^x - 2x e^x + 2e^x]_1^2 \simeq 12.06,$$

using integration by parts (twice).

The area is given by

$$\int_1^2 x e^x \, dx = [x e^x - e^x]_1^2 \simeq 7.39$$

using integration by parts (once).

Hence,

$$\bar{x} \simeq 12.06 \div 7.39 \simeq 1.63$$

The first moment about the x -axis is approximately 33.20, from Example 2 in section 13.7.3; and so,

$$\bar{y} \simeq 33.20 \div 7.39 \simeq 4.47$$

13.7.5 EXERCISES

Determine the position of the centroid of each of the following regions of the xy -plane:

1. Bounded in the first quadrant by the x -axis, the y -axis and the curve whose equation is

$$x^2 + y^2 = a^2.$$

2. Bounded by the line $x = 1$ and the semi-circle whose equation is

$$(x - 1)^2 + y^2 = 4, \quad x > 1.$$

3. Bounded in the fourth quadrant by the x -axis, the y -axis and the curve whose equation is

$$y = 2x^2 - 1.$$

4. Bounded in the first quadrant by the x -axis and the curve whose equation is

$$y = \sin x.$$

5. Bounded in the first quadrant by the x -axis, the y -axis, the line $x = 1$ and the curve whose equation is

$$y = xe^{-2x}.$$

13.7.6 ANSWERS TO EXERCISES

1.

$$\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right).$$

2.

$$\left(\frac{11}{3\pi}, 0\right).$$

3.

$$\left(\frac{3\sqrt{2}}{16}, -\frac{13}{20}\right).$$

4.

$$\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

5.

$$(0.28, 0.04).$$