

“JUST THE MATHS”

UNIT NUMBER

13.6

INTEGRATION APPLICATIONS 6
(First moments of an arc)

by

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13.6.1 Introduction

13.6.2 First moment of an arc about the y -axis

13.6.3 First moment of an arc about the x -axis

13.6.4 The centroid of an arc

13.6.5 Exercises

13.6.6 Answers to exercises

UNIT 13.6 - INTEGRATION APPLICATIONS 6

FIRST MOMENTS OF AN ARC

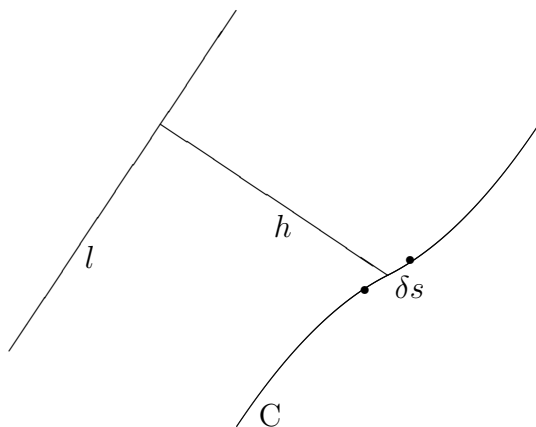
13.6.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy -plane of cartesian co-ordinates; and suppose that δs is the length of a small element of this arc.

Then the “**first moment**” of C about a fixed line, l , in the plane of C is given by

$$\lim_{\delta s \rightarrow 0} \sum_C h \delta s,$$

where h is the perpendicular distance, from l , of the element with length δs .

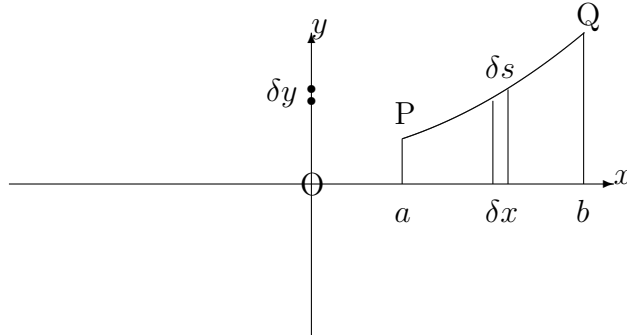


13.6.2 FIRST MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve, whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs , by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x -axis) and δy (parallel to the y -axis).

The first moment of each element about the y -axis is x times the length of the element; that is $x\delta s$, implying that the total first moment of the arc about the y -axis is given by

$$\lim_{\delta s \rightarrow 0} \sum_C x \delta s.$$

But, from Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x,$$

so that the first moment of the arc becomes

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} x \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x = \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the y -axis is given by

$$\pm \int_{t_1}^{t_2} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as $\frac{dx}{dt}$ is positive or negative.

13.6.3 FIRST MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the first moment about the x -axis will be

$$\int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the first moment of the arc about the x -axis is given by

$$\pm \int_{t_1}^{t_2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

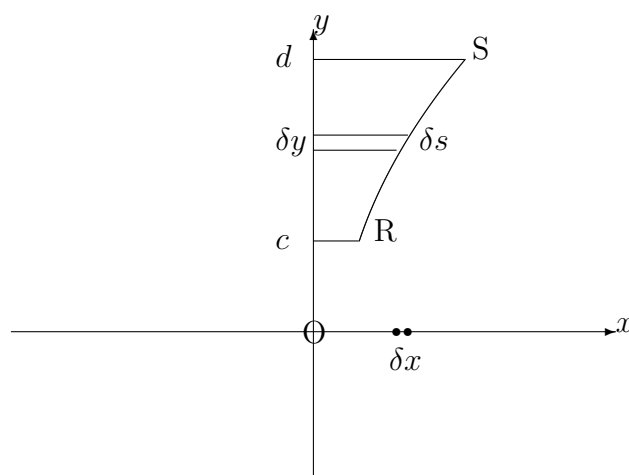
according as $\frac{dx}{dt}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.6.2 so that the first moment of the arc about the x -axis is given by

$$\int_c^d y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$



Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the x -axis is given by

$$\pm \int_{t_1}^{t_2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as $\frac{dy}{dt}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

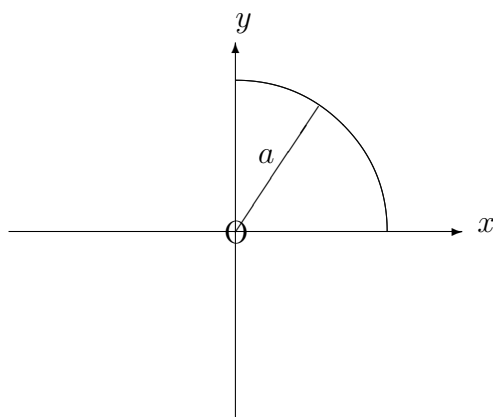
EXAMPLES

1. Determine the first moments about the x -axis and the y -axis of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y \frac{dy}{dx} = 0,$$

and hence, $\frac{dy}{dx} = -\frac{x}{y}$.

The first moment of the arc about the y -axis is therefore given by

$$\int_0^a x \sqrt{1 + \frac{x^2}{y^2}} dx = \int_0^a \frac{x}{y} \sqrt{x^2 + y^2} dx.$$

But $x^2 + y^2 = a^2$ and $y = \sqrt{a^2 - x^2}$.

Hence,

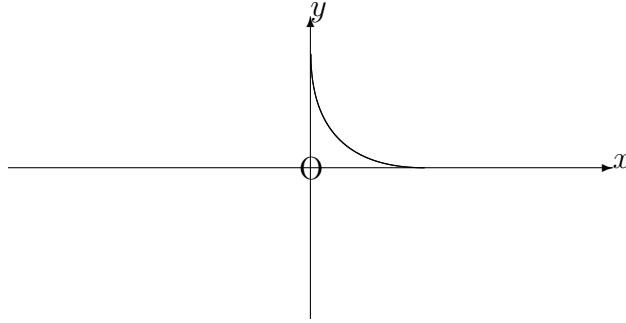
$$\text{first moment} = \int_0^a \frac{ax}{\sqrt{a^2 - x^2}} dx = \left[-a\sqrt{(a^2 - x^2)} \right]_0^a = a^2.$$

By symmetry, the first moment of the arc about the x -axis will also be a^2 .

2. Determine the first moments about the x -axis and the y -axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{dx}{d\theta} = -3a\cos^2\theta \sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\sin^2\theta \cos\theta.$$

Hence, the first moment about the x -axis is given by

$$- \int_{\frac{\pi}{2}}^0 y \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta} \, d\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a\sin^3\theta \cdot 3a \cos\theta \sin\theta \, d\theta$$

$$= 3a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos\theta \, d\theta$$

$$= 3a^2 \left[\frac{\sin^5\theta}{5} \right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}.$$

Similarly, the first moment of the arc about the y -axis is given by

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta &= \int_0^{\frac{\pi}{2}} a \cos^3 \theta \cdot (3a \cos \theta \sin \theta) d\theta \\ &= 3a^2 \int_0^{\frac{\pi}{2}} \cos^4 \theta \sin \theta d\theta = 3a^2 \left[-\frac{\cos^5 \theta}{5} \right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}, \end{aligned}$$

though, again, this second result could be deduced, by symmetry, from the first.

13.6.4 THE CENTROID OF AN ARC

Having calculated the first moments of an arc about both the x -axis and the y -axis it is possible to determine a point, (\bar{x}, \bar{y}) , in the xy -plane with the property that

(a) The first moment about the y -axis is given by $s\bar{x}$, where s is the total length of the arc; and

(b) The first moment about the x -axis is given by $s\bar{y}$, where s is the total length of the arc.

The point is called the “**centroid**” or the “**geometric centre**” of the arc and, for an arc of the curve with equation $y = f(x)$, between $x = a$ and $x = b$, its co-ordinates are given by

$$\bar{x} = \frac{\int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx} \quad \text{and} \quad \bar{y} = \frac{\int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}.$$

Notes:

(i) The first moment of an arc about an axis through its centroid will, by definition, be zero. In particular, if we take the y -axis to be parallel to the given axis, with x as the perpendicular distance from an element, δs , to the y -axis, the first moment about the given axis will be

$$\sum_C (x - \bar{x}) \delta s = \sum_C x \delta s - \bar{x} \sum_C \delta s = s\bar{x} - s\bar{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole arc at a single point for the purposes of considering first moments. In practice, it corresponds, for example, to the position of the centre of mass of a thin wire with uniform density.

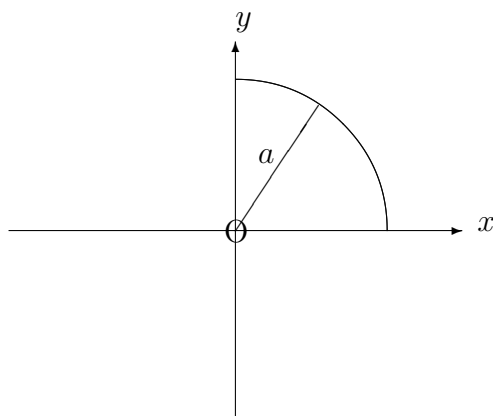
EXAMPLES

1. Determine the cartesian co-ordinates of the centroid of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an earlier example in this unit, we know that the first moments of the arc about the x -axis and the y -axis are both equal to a^2 .

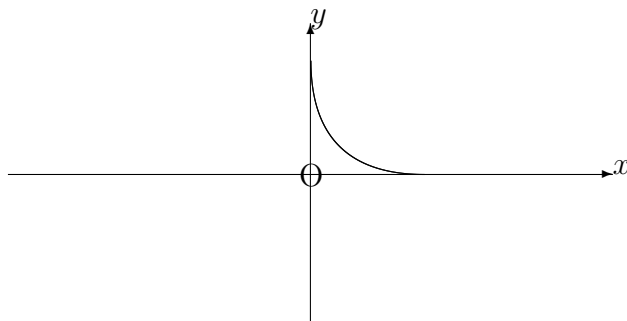
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that

$$\bar{x} = \frac{2a}{\pi} \quad \text{and} \quad \bar{y} = \frac{2a}{\pi}.$$

2. Determine the cartesian co-ordinates of the centroid of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From an earlier example in this unit, we know that

$$\frac{dx}{d\theta} = -3a\cos^2\theta \sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\sin^2\theta \cos\theta$$

and that the first moments of the arc about the x -axis and the y -axis are both equal to $\frac{3a^2}{5}$.

Also, the length of the arc is given by

$$- \int_{\frac{\pi}{2}}^a \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta} d\theta.$$

This simplifies to

$$3a \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta d\theta = 3a \left[\frac{\sin^2\theta}{2} \right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus,

$$\bar{x} = \frac{2a}{5} \quad \text{and} \quad \bar{y} = \frac{2a}{5}.$$

13.6.5 EXERCISES

1. Determine the first moment about the y -axis of the arc of the curve with equation

$$y = x^2,$$

lying between $x = 0$ and $x = 1$.

2. Determine the first moment about the x -axis of the arc of the curve with equation

$$x = 5y^2,$$

lying between $y = 0.1$ and $y = 0.5$.

3. Determine the first moment about the x -axis of the arc of the curve with equation

$$y = 2\sqrt{x},$$

lying between $x = 3$ and $x = 24$.

4. Verify, using integration, that the centroid of the straight line segment, defined by the equation

$$y = 3x + 2,$$

from $x = 0$ to $x = 1$, lies at its centre point.

5. Determine the cartesian co-ordinates of the centroid of the arc of the circle given parametrically by

$$x = 5 \cos \theta, \quad y = 5 \sin \theta,$$

from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$.

6. For the curve whose equation is

$$9y^2 = x(3 - x)^2,$$

show that

$$\frac{dy}{dx} = \frac{1 - x}{2\sqrt{x}}.$$

Hence show that the centroid of the first quadrant arch of this curve lies at the point $\left(\frac{7}{5}, \frac{\sqrt{3}}{4}\right)$.

13.6.6 ANSWERS TO EXERCISES

1.

$$\frac{5\sqrt{5} - 1}{12} \simeq 0.85$$

2.

$$\frac{13\sqrt{26} - \sqrt{2}}{150} \simeq 0.43$$

3.

156.

4.

$$\bar{x} = \frac{1}{2} \text{ and } \bar{y} = \frac{7}{2}.$$

5.

$$\bar{x} = \frac{15}{\pi} \simeq 4.77, \quad \bar{y} = 0.$$