

**“JUST THE MATHS”**

**UNIT NUMBER**

**13.5**

**INTEGRATION APPLICATIONS 5  
(Surfaces of revolution)**

by

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## UNIT 13.5 - INTEGRATION APPLICATIONS 5

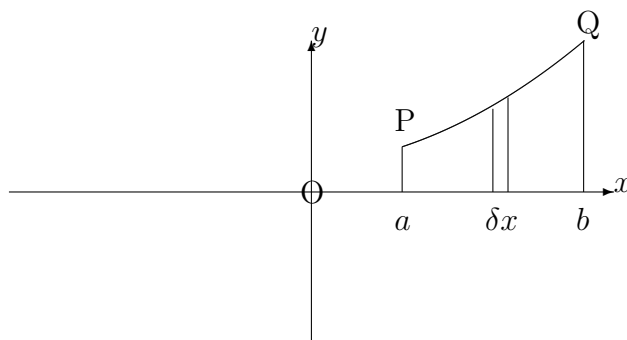
### SURFACES OF REVOLUTION

#### 13.5.1 SURFACES OF REVOLUTION ABOUT THE X-AXIS

The problem, in this unit, is to calculate the surface area obtained when the arc of the curve, with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which  $x = a$  and  $x = b$ , is rotated through  $2\pi$  radians about the  $x$ -axis or the  $y$ -axis.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of  $\delta x$  and  $\delta y$ , parallel to the  $x$ -axis and the  $y$ -axis, respectively, then the length,  $\delta s$ , of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x,$$

using Pythagoras's Theorem.

When the arc, of length  $\delta s$ , is rotated through  $2\pi$  radians about the  $x$ -axis, it generates a thin band whose area is, approximately,

$$2\pi y \delta s = 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

The total surface area,  $S$ , is thus given by

$$S = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

That is,

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

**Note:**

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Hence,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\frac{dx}{dt}},$$

provided  $\frac{dx}{dt}$  is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{t_1}^{t_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dt} dt,$$

where  $t = t_1$  when  $x = a$  and  $t = t_2$  when  $x = b$ .

We may conclude that

$$S = \pm \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as  $\frac{dx}{dt}$  is positive or negative.

### EXAMPLES

1. A curve has equation

$$y^2 = 2x.$$

Determine the surface area obtained when the arc of the curve between the point (2, 2) and the point (8, 4) is rotated through  $2\pi$  radians about the  $x$ -axis.

#### Solution

We may write the equation of the arc of the curve in the form

$$y = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}};$$

and so,

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2x}}.$$

Hence,

$$S = \int_2^8 2\pi\sqrt{2x}\sqrt{1 + \frac{1}{2x}} dx = \int_2^8 \sqrt{2x+1} dx = \left[\frac{(2x+1)^{\frac{3}{2}}}{3}\right]_2^8.$$

Thus,

$$S = \frac{17^{\frac{3}{2}}}{3} - \frac{5^{\frac{3}{2}}}{3} \simeq 19.64$$

2. A curve is given parametrically by

$$x = \sqrt{2} \cos \theta, \quad y = \sqrt{2} \sin \theta.$$

Determine the surface area obtained when the arc of the curve between the point  $(0, \sqrt{2})$  and the point  $(1, 1)$  is rotated through  $2\pi$  radians about the  $x$ -axis.

**Solution**

The parameters of the two points are  $\frac{\pi}{2}$  and  $\frac{\pi}{4}$ , respectively; and, since

$$\frac{dx}{d\theta} = -\sqrt{2} \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = \sqrt{2} \cos \theta,$$

we have

$$S = - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\sqrt{2}\pi \sin \theta \sqrt{2\sin^2 \theta + 2\cos^2 \theta} \, d\theta = - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\pi \sin \theta \, d\theta.$$

Thus,

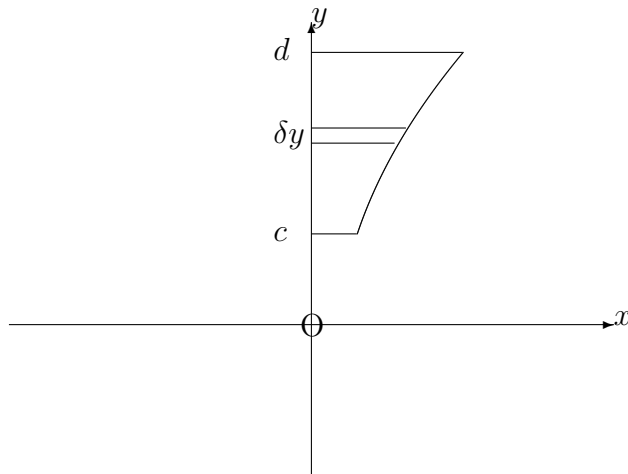
$$S = - [-4\pi \cos \theta]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{4\pi}{\sqrt{2}} \simeq 8.89$$

### 13.5.2 SURFACES OF REVOLUTION ABOUT THE Y-AXIS

For a curve whose equation is of the form  $x = g(y)$ , the surface of revolution about the  $y$ -axis of an arc joining the two points at which  $y = c$  and  $y = d$  is given by

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.$$

We simply reverse the roles of  $x$  and  $y$  in the previous section.



Alternatively, if the curve is given parametrically,

$$S = \pm \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as  $\frac{dy}{dt}$  is positive or negative.

### EXAMPLE

If the arc of the parabola, with equation

$$x^2 = 2y,$$

joining the two points (2, 2) and (4, 8), is rotated through  $2\pi$  radians about the  $y$ -axis, determine the surface area obtained.

### Solution

Using the result from the previous section, the surface area obtained is given by

$$S = \int_2^8 2\pi\sqrt{2y}\sqrt{1 + \frac{1}{2y}} dy \simeq 19.64$$

### 13.5.3 EXERCISES

1. Use a straight line through the origin to determine the surface area of a right-circular cone with height,  $h$ , and base radius,  $r$ .
2. Determine the surface area obtained when the arc of the curve  $x = y^3$ , between  $y = 0$  and  $y = 1$ , is rotated through  $2\pi$  radians about the  $y$ -axis.
3. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

Determine the surface area obtained when the arc of the curve between the point where  $t = 0$  and the point where  $t = \frac{\pi}{2}$  is rotated through  $2\pi$  radians about the  $x$ -axis.

State your answer correct to three places of decimals.

4. A curve is given parametrically by

$$x = 4(\cos \theta + \theta \sin \theta), \quad y = 4(\sin \theta - \theta \cos \theta).$$

Determine the surface area obtained when the arc of the curve between the point where  $\theta = 0$  and the point where  $\theta = \frac{\pi}{2}$  is rotated through  $2\pi$  radians about the  $x$ -axis.

5. A curve is given parametrically by

$$x = e^u \cos u, \quad y = e^u \sin u.$$

Determine the surface area obtained when the arc of the curve between the point where  $u = 0$  and the point where  $u = \frac{\pi}{4}$  is rotated through  $2\pi$  radians about the  $y$ -axis.

State your answer correct to three places of decimals.

### 13.5.4 ANSWERS TO EXERCISES

1.

$$\pi r \sqrt{r^2 + h^2}.$$

2.

$$\frac{\pi(10\sqrt{10} - 1)}{27} \simeq 3.56$$

3.

$$3.891$$

4.

$$32\pi \left( 3 - \left( \frac{\pi}{2} \right)^2 \right) \simeq 53.54$$

5.

$$1.037$$