

**“JUST THE MATHS”**

**UNIT NUMBER**

**13.2**

**INTEGRATION APPLICATIONS 2**

**(Mean values)**

**&**

**(Root mean square values)**

**by**

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**13.2.1 Mean values**

**13.2.2 Root mean square values**

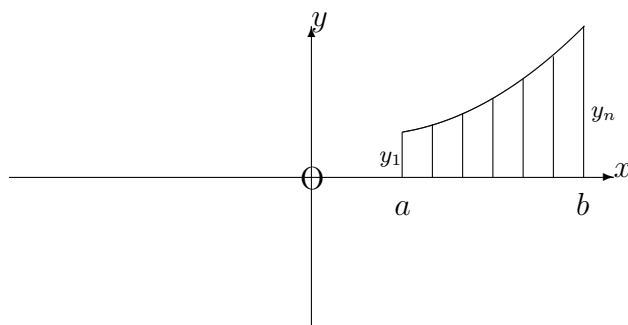
**13.2.3 Exercises**

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## UNIT 13.2 - INTEGRATION APPLICATIONS 2

### MEAN AND ROOT MEAN SQUARE VALUES

#### 13.2.1 MEAN VALUES



On the curve whose equation is

$$y = f(x),$$

suppose that  $y_1, y_2, y_3, \dots, y_n$  are the  $y$ -coordinates which correspond to  $n$  different  $x$ -coordinates,  $a = x_1, x_2, x_3, \dots, x_n = b$ .

The average (that is, the arithmetic mean) of these  $n$   $y$ -coordinates is

$$\frac{y_1 + y_2 + y_3 + \dots + y_n}{n}.$$

But now suppose that we wished to determine the average (arithmetic mean) of **all** the  $y$ -coordinates, from  $x = a$  to  $x = b$  on the curve whose equation is  $y = f(x)$ .

We could make a reasonable approximation by taking a very **large** number,  $n$ , of  $y$ -coordinates separated in the  $x$ -direction by very **small** distances. If these distances are typically represented by  $\delta x$ , then the required mean value could be written

$$\frac{y_1\delta x + y_2\delta x + y_3\delta x + \dots + y_n\delta x}{n\delta x},$$

in which the denominator is equivalent to  $(b - a + \delta x)$ , since there are only  $n - 1$  spaces between the  $n$   $y$ -coordinates.

Allowing the number of  $y$ -coordinates to increase indefinitely,  $\delta x$  will tend to zero and we obtain the formula for the “**Mean Value**” in the form

$$\text{M.V.} = \frac{1}{b - a} \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} y \delta x.$$

That is,

$$\text{M.V.} = \frac{1}{b - a} \int_a^b f(x) \, dx.$$

**Note:**

In cases where the definite integral in this formula represents the area between the curve and the  $x$ -axis, the Mean Value provides the height of a rectangle, with base  $b - a$ , having the same area as that represented by the definite integral.

**EXAMPLE**

Determine the Mean Value of the function

$$f(x) \equiv x^2 - 5x$$

from  $x = 1$  to  $x = 4$ .

**Solution**

The Mean Value is given by

$$\text{M.V.} = \frac{1}{4 - 1} \int_1^4 (x^2 - 5x) \, dx = \frac{1}{3} \left[ \frac{x^3}{3} - \frac{5x^2}{2} \right]_1^4 =$$

$$\frac{1}{3} \left[ \left( \frac{64}{3} - 40 \right) - \left( \frac{1}{3} - \frac{5}{2} \right) \right] = -\frac{33}{2}.$$

### 13.2.2 ROOT MEAN SQUARE VALUES

It is sometimes convenient to use an alternative kind of average for the values of a function,  $f(x)$ , between  $x = a$  and  $x = b$ .

The “**Root Mean Square Value**” provides a measure of “central tendency” for the **numerical** values of  $f(x)$  and is defined to be the square root of the Mean Value of  $f(x)$  from  $x = a$  to  $x = b$ .

Hence,

$$\text{R.M.S.V.} = \sqrt{\frac{1}{b-a} \int_a^b [f(x)]^2 dx}.$$

#### EXAMPLE

Determine the Root Mean Square Value of the function,  $f(x) \equiv x^2 - 5$ , from  $x = 1$  to  $x = 3$ .

#### Solution

The Root Mean Square Value is given by

$$\text{R.M.S.V.} = \sqrt{\frac{1}{3-1} \int_1^3 (x^2 - 5)^2 dx}.$$

Temporarily ignoring the square root, we obtain the “**Mean Square Value**”,

$$\text{M.S.V.} = \frac{1}{2} \int_1^3 (x^4 - 10x^2 + 25) dx$$

$$= \frac{1}{2} \left[ \frac{x^5}{5} - \frac{10x^3}{3} + 25x \right]_1^3 = \frac{1}{2} \left[ \left( \frac{243}{5} - \frac{270}{3} + 75 \right) - \left( \frac{1}{5} - \frac{10}{3} + 25 \right) \right] = \frac{176}{30}.$$

Thus,

$$\text{R.M.S.V.} = \sqrt{\frac{176}{30}} \simeq 2.422$$

### 13.2.3 EXERCISES

- Determine the Mean Value of the function,  $(x - 1)(x - 2)$ , from  $x = 1$  to  $x = 2$ ;
  - Determine, correct to three significant figures, the Mean Value of the function,  $\frac{1}{2x+5}$ , from  $x = 3$  to  $x = 5$ ;
  - Determine the Mean Value of the function,  $\sin 2t$ , from  $t = 0$  to  $t = \frac{\pi}{2}$ ;
  - Determine, correct to three places of decimals, the Mean Value of the function,  $e^{-x}$ , from  $x = 1$  to  $x = 5$ ;
  - Determine, correct to three significant figures, the mean value of the function,  $xe^{-2x}$ , from  $x = 0$  to  $x = 2$ .
- Determine the Root Mean Square Value of the function,  $3x + 1$ , from  $x = -2$  to  $x = 2$ ;
  - Determine the Root Mean Square Value, of the function,  $e^x$ , from  $x = 0$  to  $x = 1$ , correct to three decimal places;
  - Determine the Root Mean Square Value of the function,  $\cos x$ , from  $x = \frac{\pi}{2}$  to  $x = \pi$ ;
  - Determine the Root Mean Square Value of the function,  $(4x - 5)^{\frac{3}{2}}$ , from  $x = 1.25$  to  $x = 1.5$ .

### 13.2.4 ANSWERS TO EXERCISES

- $-\frac{1}{6}$ ;
  - 0.0775;
  - $\frac{2}{\pi}$ ;
  - 0.076;
  - 0.114
- $\sqrt{13} \simeq 3.606$ ;
  - 1.787;
  - $\frac{1}{\sqrt{2}}$ ;
  - $\frac{1}{2}$ .