"JUST THE MATHS"

UNIT NUMBER

13.12

INTEGRATION APPLICATIONS 12
(Second moments of an area (B))

by

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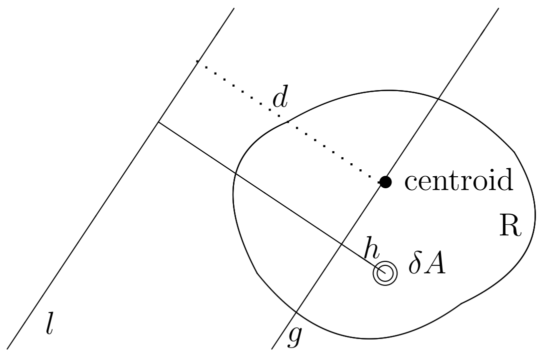
UNIT 13.12 - INTEGRATION APPLICATIONS 12

SECOND MOMENTS OF AN AREA (B)

13.12.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis, in the same plane as R and having a perpendicular distance of d from the first axis.



We have

$$M_l = \sum_{\text{R}} (h+d)^2 \delta A = \sum_{\text{R}} (h^2 + 2hd + d^2).$$

That is,

$$M_l = \sum_{\text{R}} h^2 \delta A + 2d \sum_{\text{R}} h\delta A + d^2 \sum_{\text{R}} \delta A = M_g + Ad^2,$$

since the summation, $\sum_{\text{R}} h\delta A$, is the first moment about the an axis through the centroid and therefore zero; (see Unit 13.7, section 13.7.4).

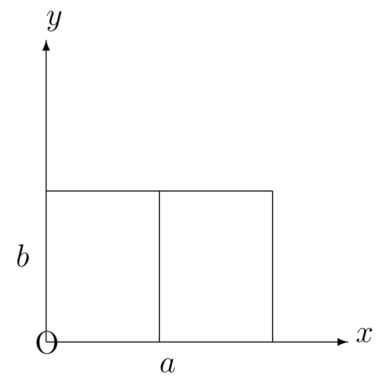
The Parallel Axis Theorem states that

$$M_l = M_g + Ad^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region about an axis through its centroid, parallel to one side.

   Solution

   For a rectangular region with sides of length a and b, the second moment about the side of length b is $\frac{a^3 b}{3}$ from Example 1 in the previous Unit, section 13.11.2.

   The perpendicular distance between the two axes is then $\frac{a}{2}$, so that the required second moment, M_g is given by

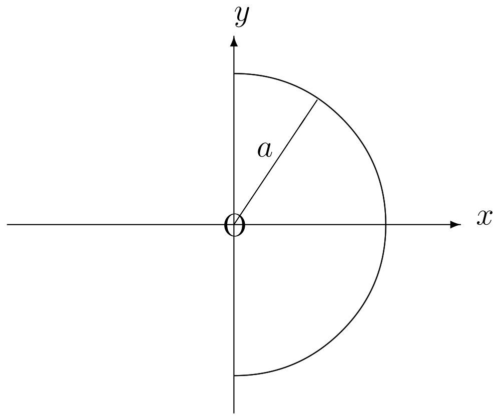
   $$\frac{a^3 b}{3} = M_g + ab\left(\frac{a}{2}\right)^2 = M_g + \frac{a^3 b}{4}$$

   Hence,

   $$M_g = \frac{a^3 b}{12}.$$

2. Determine the second moment of a semi-circular region about an axis through its centroid, parallel to its diameter.

Solution



The second moment of the semi-circular region about its diameter is $\frac{\pi a^4}{8}$, from Example 2 in the previous Unit, section 13.11.2.

Also the position of the centroid, from Example 2 in Unit 13.7, section 13.7.4, is a distance of $\frac{4a}{3\pi}$ from the diameter, along the radius which perpendicular to it.

Hence,

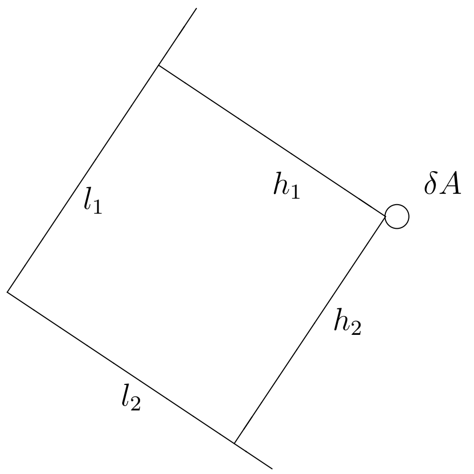
$$\frac{\pi a^4}{8} = M_g + \frac{\pi a^2}{2}.\left(\frac{4a}{3\pi}\right)^2 = M_g + \frac{8a^4}{9\pi^2}.$$

That is,

$$M_g = \frac{\pi a^4}{8} - \frac{8a^4}{9\pi^2}.$$

13.12.2 THE PERPENDICULAR AXIS THEOREM

Suppose l_1 and l_2 are two straight lines, at right-angles to each other, in the plane of a region R with area A and suppose h_1 and h_2 are the perpendicular distances from these two lines, respectively, of an element δA in R.



The second moment about l_1 is given by

$$M_1 = \sum_R h_1^2 \delta A$$

and the second moment about l_2 is given by

$$M_2 = \sum_R h_2^2 \delta A.$$

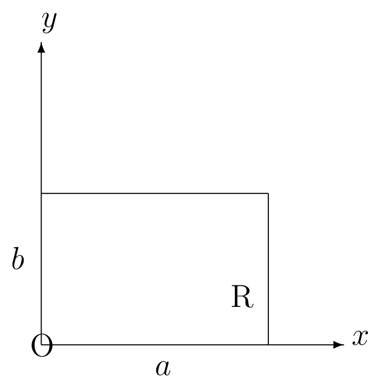
Adding these two together gives the second moment about an axis, perpendicular to the plane of R and passing through the point of intersection of l_1 and l_2. This is because the square of the perpendicular distance, h_3, of δA from this new axis is given, from Pythagoras's Theorem, by

$$h_3^2 = h_1^2 + h_2^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region, R, with sides of length a and b, about an axis through one corner, perpendicular to the plane of R.

Solution

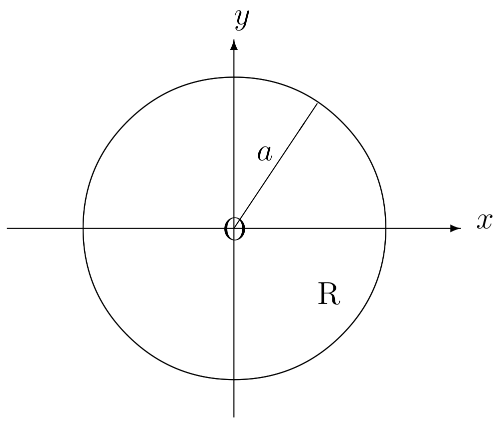


Using Example 1 in the previous Unit, section 13.11.2, the required second moment is

$$\frac{1}{3}a^3b + \frac{1}{3}b^3a = \frac{1}{3}ab(a^2 + b^2).$$

2. Determine the second moment of a circular region, R, with radius a, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment of R about a diameter is, from Example 2 in the previous Unit, section 13.11.2, equal to $\frac{\pi a^4}{4}$; that is, twice the value of the second moment of a semi-circular region about its diameter.

The required second moment is thus

$$\frac{\pi a^4}{4} + \frac{\pi a^4}{4} = \frac{\pi a^4}{2}.$$

13.12.3 THE RADIUS OF GYRATION OF AN AREA

Having calculated the second moment of a two dimensional region about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Ak^2, where A is the total area of the region.

We simply divide the value of the second moment by A in order to obtain the value of k^2 and hence the value of k.

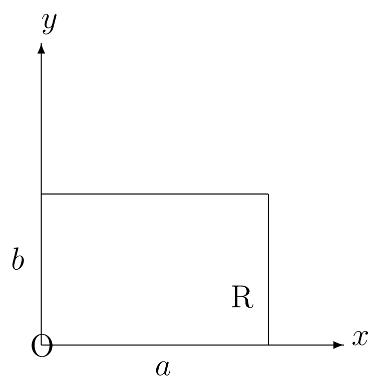
The value of k is called the **"radius of gyration"** of the given region about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole area at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a rectangular region, R, with sides of lengths a and b about an axis through one corner, perpendicular to the plane of R.

   Solution



Using Example 1 from the previous section, the second moment is

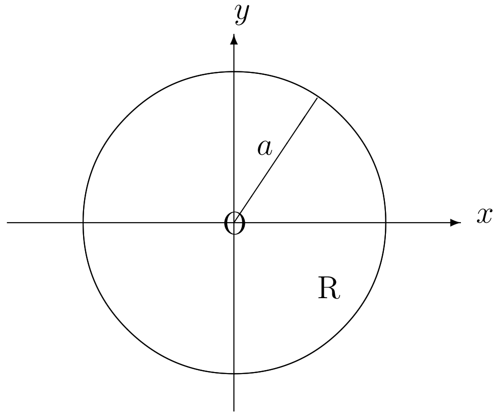
$$\frac{1}{3}ab(a^2 + b^2)$$

and, since the area itself is ab, we obtain

$$k = \sqrt{a^2 + b^2}.$$

2. Determine the radius of gyration of a circular region, R, about an axis through its centre, perpendicular to the plane of R.

Solution



From Example 2 in the previous section, the second moment about the given axis is $\frac{\pi a^4}{2}$ and, since the area itself is πa^2, we obtain

$$k = \frac{a}{\sqrt{2}}.$$

13.12.4 EXERCISES

Determine the radius of gyration of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$.
   Axis: Through the point $\left(\frac{a}{2}, \frac{b}{2}\right)$, perpendicular to the xy-plane.
2. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$.
   Axis: The line $x = 4a$.
3. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$x^2 + y^2 = a^2.$$
   Axis: Through the origin, perpendicular to the xy-plane.
4. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$x^2 + y^2 = a^2.$$
   Axis: The line $x = a$.

13.12.5 ANSWERS TO EXERCISES

1. $$\frac{1}{12}\left(a^2 + b^2\right).$$
2. $$\frac{7a}{\sqrt{3}}.$$
3. $$\frac{a}{\sqrt{2}}.$$
4. $$\frac{a\sqrt{5}}{2}.$$