

**“JUST THE MATHS”**

**UNIT NUMBER**

**12.8**

**INTEGRATION 8**  
**(The tangent substitutions)**

by

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**12.8.1** The substitution  $t = \tan x$

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## UNIT 12.8 - INTEGRATION 8

### THE TANGENT SUBSTITUTIONS

There are two types of integral, involving sines and cosines, which require a special substitution using a tangent function. They are described as follows:

#### 12.8.1 THE SUBSTITUTION $t = \tan x$

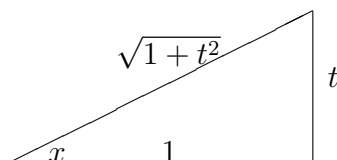
This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin^2 x + c\cos^2 x} dx,$$

where  $a$ ,  $b$  and  $c$  are constants; though, in most exercises, at least one of these three constants will be zero.

A simple right-angled triangle will show that, if  $t = \tan x$ , then

$$\sin x \equiv \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos x \equiv \frac{1}{\sqrt{1+t^2}}.$$



Furthermore,

$$\frac{dt}{dx} \equiv \sec^2 x \equiv 1 + t^2 \quad \text{so that} \quad \frac{dx}{dt} \equiv \frac{1}{1+t^2}.$$

### EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{4 - 3\sin^2 x} dx.$$

**Solution**

$$\begin{aligned} & \int \frac{1}{4 - 3\sin^2 x} dx \\ &= \int \frac{1}{4 - \frac{3t^2}{1+t^2}} \cdot \frac{1}{1+t^2} dt \\ &= \int \frac{1}{4+t^2} dt \\ &= \frac{1}{2} \tan^{-1} \frac{t}{2} + C = \frac{1}{2} \tan^{-1} \left[ \frac{\tan x}{2} \right] + C. \end{aligned}$$

2. Determine the indefinite integral

$$\int \frac{1}{\sin^2 x + 9\cos^2 x} dx.$$

**Solution**

$$\begin{aligned} & \int \frac{1}{\sin^2 x + 9\cos^2 x} dx \\ &= \int \frac{1}{\frac{t^2}{1+t^2} + \frac{9}{1+t^2}} \cdot \frac{1}{1+t^2} dt \\ &= \int \frac{1}{t^2 + 9} dt \\ &= \frac{1}{3} \tan^{-1} \frac{t}{3} + C = \frac{1}{3} \tan^{-1} \left[ \frac{\tan x}{3} \right] + C. \end{aligned}$$

### 12.8.2 THE SUBSTITUTION $t = \tan(x/2)$

This substitution is used for integrals of the form

$$\int \frac{1}{a + b \sin x + c \cos x} dx,$$

where  $a$ ,  $b$  and  $c$  are constants; though, in most exercises, one or more of these constants will be zero.

In order to make the substitution, we make the following observations:

(i)

$$\sin x \equiv 2 \sin(x/2) \cdot \cos(x/2) \equiv 2 \tan(x/2) \cdot \cos^2(x/2) \equiv \frac{2 \tan(x/2)}{\sec^2(x/2)} \equiv \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}.$$

Hence,

$$\sin x \equiv \frac{2t}{1 + t^2}.$$

(ii)

$$\cos x \equiv \cos^2(x/2) - \sin^2(x/2) \equiv \cos^2(x/2) [1 - \tan^2(x/2)] \equiv \frac{1 - \tan^2(x/2)}{\sec^2(x/2)} \equiv \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}.$$

Hence,

$$\cos x \equiv \frac{1 - t^2}{1 + t^2}.$$

(iii)

$$\frac{dt}{dx} \equiv \frac{1}{2} \sec^2(x/2) \equiv \frac{1}{2} [1 + \tan^2(x/2)] \equiv \frac{1}{2} [1 + t^2].$$

Hence,

$$\frac{dx}{dt} \equiv \frac{2}{1 + t^2}.$$

## EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{1 + \sin x} dx$$

**Solution**

$$\begin{aligned} & \int \frac{1}{1 + \sin x} \, dx \\ &= \int \frac{1}{1 + \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt \\ &= \int \frac{2}{1+t^2+2t} \, dt \\ &= \int \frac{2}{(1+t)^2} \, dt \\ &= -\frac{2}{1+t} + C = -\frac{2}{1 + \tan(x/2)} + C. \end{aligned}$$

2. Determine the indefinite integral

$$\int \frac{1}{4 \cos x - 3 \sin x} \, dx.$$

**Solution**

$$\begin{aligned} & \int \frac{1}{4 \cos x - 3 \sin x} \, dx \\ &= \int \frac{1}{4 \frac{1-t^2}{1+t^2} - \frac{6t}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt \\ &= \int \frac{2}{4 - 4t^2 - 6t} \, dt = \int -\frac{1}{2t^2 + 3t - 2} \, dt \\ &= \int -\frac{1}{(2t-1)(t+2)} \, dt \\ &= \int \frac{1}{5} \left[ \frac{1}{t+2} - \frac{2}{2t-1} \right] \, dt \\ &= \frac{1}{5} [\ln(t+2) - \ln(2t-1)] + C = \frac{1}{5} \ln \left[ \frac{\tan(x/2) + 2}{2 \tan(x/2) - 1} \right] + C. \end{aligned}$$

### 12.8.3 EXERCISES

1. Determine the indefinite integral

$$\int \frac{1}{4 + 12\cos^2 x} dx.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x + 3\sin^2 x} dx.$$

3. Determine the indefinite integral

$$\int \frac{1}{5 + 3\cos x} dx.$$

4. Evaluate the definite integral

$$\int_3^{3.1} \frac{1}{12\sin x + 5\cos x} dx.$$

### 12.8.4 ANSWERS TO EXERCISES

- 1.

$$\frac{1}{4}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

- 2.

$$\left[\frac{1}{\sqrt{15}}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right)\right]_0^{\frac{\pi}{4}} \simeq 0.1702$$

- 3.

$$\frac{1}{2}\tan^{-1}\left[\frac{\tan(x/2)}{2}\right] + C.$$

- 4.

$$\left[\frac{1}{13}[5\ln(5\tan(x/2) + 1) - \ln(\tan(x/2) - 5)]\right]_3^{3.1} \simeq 0.348$$