

“JUST THE MATHS”

UNIT NUMBER

12.6

INTEGRATION 6

(Integration by partial fractions)

by

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12.6.1 Introduction and illustrations

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UNIT 12.6 - INTEGRATION 6

INTEGRATION BY PARTIAL FRACTIONS

12.6.1 INTRODUCTION AND ILLUSTRATIONS

If the ratio of two polynomials, whose denominator has been factorised, is expressed as a sum of partial fractions, each partial fraction will be of a type whose integral can be determined by the methods of preceding sections of this chapter.

The following summary of results will cover most elementary problems involving partial fractions:

RESULTS

1.

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \ln(ax + b) + C.$$

2.

$$\int \frac{1}{(ax + b)^n} dx = \frac{1}{a} \cdot \frac{(ax + b)^{-n+1}}{-n + 1} + C \text{ provided } n \neq 1.$$

3.

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

4.

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C,$$

or

$$\frac{1}{2a} \ln \left(\frac{a + x}{a - x} \right) + C \text{ when } |x| < a,$$

and

$$\frac{1}{2a} \ln \left(\frac{x + a}{x - a} \right) + C \text{ when } |x| > a.$$

5.

$$\int \frac{2ax + b}{ax^2 + bx + c} dx = \ln(ax^2 + bx + c) + C.$$

ILLUSTRATIONS

We use some of the results of examples on partial fractions in Unit 1.8

1.

$$\begin{aligned}\int \frac{7x+8}{(2x+3)(x-1)} dx &= \int \left[\frac{1}{2x+3} + \frac{3}{x-1} \right] dx \\ &= \frac{1}{2} \ln(2x+3) + 3 \ln(x-1) + C.\end{aligned}$$

2.

$$\begin{aligned}\int_6^8 \frac{3x^2+9}{(x-5)(x^2+2x+7)} dx &= \int_6^8 \left[\frac{2}{x-5} + \frac{x+1}{x^2+2x+7} \right] dx \\ &= \left[2 \ln(x-5) + \frac{1}{2} \ln(x^2+2x+7) \right]_6^8 \simeq 2.427\end{aligned}$$

3.

$$\begin{aligned}\int \frac{9}{(x+1)^2(x-2)} &= \int \left[\frac{-1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2} \right] dx \\ &= -\ln(x+1) + \frac{3}{x+1} + \ln(x-2) + C.\end{aligned}$$

4.

$$\begin{aligned}\int \frac{4x^2+x+6}{(x-4)(x^2+4x+5)} dx &= \int \left[\frac{2}{x-4} + \frac{2x+1}{x^2+4x+5} \right] dx \\ &= 2 \ln(x-4) + \ln(x^2+4x+5) - 3 \tan^{-1}(x+2) + C.\end{aligned}$$

Note:

In the last example above, the second partial fraction has a numerator of $2x+1$ which is not the derivative of x^2+4x+5 . But we simply rearrange the numerator as $(2x+4)-3$ to give a third integral which requires the technique of completing the square (discussed in Unit 12.3).

12.6.2 EXERCISES

Integrate the following functions with respect to x :

1. (a)

$$\frac{3x + 5}{(x + 1)(x + 2)};$$

(b)

$$\frac{17x + 11}{(x + 1)(x - 2)(x + 3)};$$

(c)

$$\frac{3x^2 - 8}{(x - 1)(x^2 + x - 7)}.$$

(d)

$$\frac{2x + 1}{(x + 2)^2(x - 3)};$$

(e)

$$\frac{9 + 11x - x^2}{(x + 1)^2(x + 2)};$$

(f)

$$\frac{x^5}{(x + 2)(x - 4)}.$$

2. Evaluate the following definite integrals

(a)

$$\int_2^5 \frac{7x^2 + 11x + 47}{(x - 1)(x^2 + 2x + 10)} dx;$$

(b)

$$\int_1^3 \frac{4x^2 + 1}{x(2x - 1)^2} dx.$$

12.6.3 ANSWERS TO EXERCISES

1. (a)

$$2\ln(x+1) + \ln(x+2) + C;$$

(b)

$$\ln(x+1) + 3\ln(x-2) - 4\ln(x+3) + C;$$

(c)

$$\ln(x-1) + \ln(x^2 + x - 7) + C;$$

(d)

$$-\frac{3}{5(x+2)} - \frac{7}{25}\ln(x+2) + \frac{7}{25}\ln(x-3) + C;$$

(e)

$$-\frac{3}{(x+1)^2} + \frac{16}{x+1} - \frac{17}{x+2}$$
$$\frac{3}{x+1} + 16\ln(x+1) - 17\ln(x+2) + C;$$

(f)

$$\frac{x^4}{4} + \frac{2x^3}{3} + 6x^2 + 40x + \frac{16}{3}\ln(x+2) + \frac{512}{3}\ln(x-4) + C.$$

2. (a)

$$\left[5\ln(x-1) + \ln(x^2 + 2x + 10) + \frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_2^5 \simeq -2.726;$$

(b)

$$\left[\ln x - \frac{2}{2x-1}\right]_1^3 \simeq 2.699$$