"JUST THE MATHS"

UNIT NUMBER

12.4

INTEGRATION 4
(Integration by substitution in general)

by

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UNIT 12.4 - INTEGRATION 4
INTEGRATION BY SUBSTITUTION IN GENERAL

12.4.1 EXAMPLES USING THE STANDARD FORMULA

With any integral

$$\int f(x)\mathrm{d}x,$$

it may be convenient to make some kind of substitution relating the variable, x, to a new variable, u. In such cases, we may use the formula discussed in Unit 12.1, namely

$$\int f(x)\mathrm{d}x = \int f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u,$$

where it is assumed that, on the right hand side, the integrand has been expressed wholly in terms of u.

For this Unit, substitutions other than linear ones will be given in the problems to be solved.

EXAMPLES

1. Use the substitution $x = a\sin u$ to show that

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$$

Solution

To be precise, we shall assume for simplicity that u is the **acute** angle for which $x = a\sin u$. In effect, we shall be making the substitution $u = \sin^{-1}\frac{x}{a}$ using the principal value of the inverse function; we can certainly do this because the expression $\sqrt{a^2 - x^2}$ requires that $-a < x < a$.

If $x = a\sin u$, then $\frac{\mathrm{d}x}{\mathrm{d}u} = a\cos u$, so that the integral becomes

$$\int \frac{a\cos u}{\sqrt{a^2 - a^2\sin^2 u}}\mathrm{d}u.$$

But, from trigonometric identities,

$$\sqrt{a^2 - a^2\sin^2 u} \equiv a\cos u,$$

both sides being positive when u is an acute angle.

We are thus left with

$$\int 1\mathrm{d}u = u + C = \sin^{-1}\frac{x}{a} + C.$$

2. Use the substitution $u = \frac{1}{x}$ to determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{x\sqrt{1+x^2}}.$$

Solution

Converting the substitution to the form

$$x = \frac{1}{u},$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}.$$

Hence,

$$z = \int \frac{1}{\frac{1}{u}\sqrt{1+\frac{1}{u^2}}}. -\frac{1}{u^2}\mathrm{d}u$$

That is,

$$z = \int -\frac{1}{\sqrt{u^2+1}} = -\ln(u + \sqrt{u^2+1}) + C.$$

Returning to the original variable, x, we have

$$z = -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) + C.$$

Note:

This example is somewhat harder than would be expected under examination conditions.

12.4.2 INTEGRALS INVOLVING A FUNCTION AND ITS DERIVATIVE

The method of integration by substitution provides two useful results applicable to a wide range of problems. They are as follows:

(a)

$$\int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

provided $n \neq -1$.

(b)

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x) + C.$$

These two results are readily established by means of the substitution

$$u = f(x).$$

In both cases $\frac{\mathrm{d}u}{\mathrm{d}x} = f'(x)$ and hence $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{f'(x)}$. This converts the integrals, respectively, into

(a)

$$\int u^n \mathrm{d}u = \frac{u^{n+1}}{n+1} + C$$

and (b)

$$\int \frac{1}{u} \mathrm{d}u = \ln u + C.$$

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \sin^3 x. \cos x \ \mathrm{d}x.$$

   Solution

   In this example we can consider $\sin x$ to be $f(x)$ and $\cos x$ to be $f'(x)$.

   Thus, by quoting result (a), we obtain

$$\int_0^{\frac{\pi}{3}} \sin^3 x. \cos x \mathrm{d}x = \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{3}} = \frac{9}{64},$$

   using $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

2. Integrate the function

$$\frac{2x+1}{x^2+x-11}$$

   with respect to x.

   Solution

   Here, we can identify $x^2 + x - 11$ with $f(x)$ and $2x + 1$ with $f'(x)$.

   Thus, by quoting result (b), we obtain

$$\int \frac{2x+1}{x^2+x-11} \mathrm{d}x = \ln(x^2 + x - 11) + C.$$

12.4.3 EXERCISES

1. Use the substitution $u = x + 3$ in order to determine the indefinite integral

$$\int x\sqrt{3+x}\ \mathrm{d}x.$$

2. Use the substitution $u = x^2 - 1$ in order to evaluate the definite integral

$$\int_1^5 x\sqrt{x^2-1}\ \mathrm{d}x.$$

3. Integrate the following functions with respect to x:

   (a)

$$\sin^7 x.\cos x;$$

   (b)

$$\cos^5 x.\sin x;$$

   (c)

$$\frac{4x-3}{2x^2-3x+13};$$

   (d)

$$\cot x.$$

12.4.4 ANSWERS TO EXERCISES

1.

$$\frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.$$

2.

$$\left[\frac{1}{3}(x^2-1)^{\frac{3}{2}}\right]_1^5 = \frac{1}{3}24^{\frac{3}{2}} \simeq 39.192$$

3. (a)

$$\frac{\sin^8 x}{8} + C;$$

(b)

$$-\frac{\cos^6 x}{6} + C;$$

(c)

$$\ln(2x^2 - 3x + 13) + C;$$

(d)

$$\ln \sin x + C.$$