

**“JUST THE MATHS”**

**UNIT NUMBER**

**12.1**

**INTEGRATION 1**

**(Elementary indefinite integrals)**

**by**

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## UNIT 12.1 - INTEGRATION 1 - ELEMENTARY INDEFINITE INTEGRALS

### 12.1.1 THE DEFINITION OF AN INTEGRAL

In Differential Calculus, we are given functions of  $x$  and asked to obtain their derivatives; but, in Integral Calculus, we are given functions of  $x$  and asked what they are the derivatives of. The process of answering this question is called “**integration**”.

In other words **integration is the reverse of differentiation**.

#### DEFINITION

Given a function  $f(x)$ , another function,  $z$ , such that

$$\frac{dz}{dx} = f(x)$$

is called an integral of  $f(x)$  with respect to  $x$ .

#### Notes:

(i) The above definition refers to **an** integral of  $f(x)$  rather than **the** integral of  $f(x)$ . This is because, having found a possible function,  $z$ , such that

$$\frac{dz}{dx} = f(x),$$

$z + C$  is also an integral for any constant value,  $C$ .

(ii) We call  $z + C$  the “**indefinite integral of  $f(x)$  with respect to  $x$** ” and we write

$$\int f(x)dx = z + C.$$

(iii)  $C$  is an **arbitrary constant** called the “**constant of integration**”.

(iv) The symbol  $dx$  does not denote a number; it is to be taken as a label indicating the variable with respect to which we are integrating. It may seem obvious that this will be  $x$ , but it could happen, for instance, that  $x$  is already dependent upon some other variable,  $t$ , in which case it would be vital to indicate the variable with respect to which we are integrating.

(v) In any integration problem, the function being integrated is called the “**integrand**”.

#### Result:

Two functions  $z_1$  and  $z_2$  are both integrals of the same function  $f(x)$  if and only if they differ by a constant.

**Proof:**

(a) Suppose, firstly, that

$$z_1 - z_2 = C,$$

where  $C$  is a constant.

Then,

$$\frac{d}{dx}[z_1 - z_2] = 0.$$

That is,

$$\frac{dz_1}{dx} - \frac{dz_2}{dx} = 0,$$

or

$$\frac{dz_1}{dx} = \frac{dz_2}{dx}.$$

From our definition, this shows that both  $z_1$  and  $z_2$  are integrals of the same function.

(b) Secondly, suppose that  $z_1$  and  $z_2$  are integrals of the same function. Then

$$\frac{dz_1}{dx} = \frac{dz_2}{dx}.$$

That is,

$$\frac{dz_1}{dx} - \frac{dz_2}{dx} = 0,$$

or

$$\frac{d}{dx}[z_1 - z_2] = 0.$$

Hence,

$$z_1 - z_2 = C,$$

where  $C$  may be any constant.

## ILLUSTRATIONS

Any result so far encountered in differentiation could be re-stated in reverse as a result on integration as shown in the following illustrations:

1.

$$\int 3x^2 dx = x^3 + C.$$

2.

$$\int x^2 dx = \frac{x^3}{3} + C.$$

3.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{Provided } n \neq -1.$$

4.

$$\int \frac{1}{x} dx \quad \text{i.e.} \quad \int x^{-1} dx = \ln x + C.$$

5.

$$\int e^x dx = e^x + C.$$

6.

$$\int \cos x dx = \sin x + C.$$

7.

$$\int \sin x dx = -\cos x + C.$$

**Note:**

Basic integrals of the above kinds may simply be quoted from a table of standard integrals in a suitable formula booklet. More advanced integrals are obtainable using the rules to be discussed below.

### 12.1.2 ELEMENTARY TECHNIQUES OF INTEGRATION

#### (a) Linearity

Suppose  $f(x)$  and  $g(x)$  are two functions of  $x$  while  $A$  and  $B$  are constants. Then,

$$\int [Af(x) + Bg(x)] dx = A \int f(x) dx + B \int g(x) dx.$$

The proof follows from the fact that differentiation is already linear and hence the derivative of the right hand side is the integrand of the left hand side. The result itself is easily extended to linear combinations of three or more functions.

#### ILLUSTRATIONS

1.

$$\int (x^2 + 3x - 7) dx = \frac{x^3}{3} + 3\frac{x^2}{2} - 7x + C.$$

2.

$$\int (3 \cos x + 4 \sec^2 x) dx = 3 \sin x + 4 \tan x + C.$$

## (b) Functions of a Linear Function

### (i) Inspection Method

Provided the method of **differentiating** functions of a linear function has been fully understood, the fastest method of **integrating** such functions is to examine, by inspection, what needs to be differentiated in order to arrive at them.

## EXAMPLES

1. Determine the indefinite integral

$$\int (2x + 3)^{12} dx.$$

### Solution

In order to arrive at the function  $(2x + 3)^{12}$  by a differentiation process, we would have to begin with a function related to  $(2x + 3)^{13}$ .

In fact,

$$\frac{d}{dx} [(2x + 3)^{13}] = 13(2x + 3)^{12} \cdot 2 = 26(2x + 3)^{12}.$$

Since this is 26 times the function we are trying to integrate, we may say that

$$\int (2x + 3)^{12} = \frac{(2x + 3)^{13}}{26} + C.$$

2. Determine the indefinite integral

$$\int \cos(3 - 5x) dx.$$

### Solution

In order to arrive at the function  $\cos(3 - 5x)$  by a differentiation process, we would have to begin with a function related to  $\sin(3 - 5x)$ .

In fact,

$$\frac{d}{dx} [\sin(3 - 5x)] = \cos(3 - 5x) \cdot -5 = -5 \cos(3 - 5x).$$

Since this is  $-5$  times the function we are trying to integrate, we may say that

$$\int \cos(3 - 5x) = -\frac{\sin(3 - 5x)}{5} + C.$$

3. Determine the indefinite integral

$$\int e^{4x+1} dx.$$

**Solution**

In order to arrive at the function  $e^{4x+1}$  by a differentiation process, we would have to begin with a function related to  $e^{4x+1}$  itself because the derivative of a power of  $e$  always contains the **same** power of  $e$ .

In fact

$$\frac{d}{dx} [e^{4x+1}] = e^{4x+1} \cdot 4.$$

Since this is 4 times the function we are trying to integrate, we may say that

$$\int e^{4x+1} dx = \frac{e^{4x+1}}{4} + C.$$

4. Determine the indefinite integral

$$\int \frac{1}{7x+3} dx.$$

**Solution**

In order to arrive at the function  $\frac{1}{7x+3}$  by a differentiation process, we would have to begin with a function related to  $\ln(7x+3)$ .

In fact,

$$\frac{d}{dx} [\ln(7x+3)] = \frac{1}{7x+3} \cdot 7 = \frac{7}{7x+3}$$

Since this is 7 times the function we are trying to integrate, we may say that

$$\int \frac{1}{7x+3} dx = \frac{\ln(7x+3)}{7} + C.$$

**Note:**

In each of these examples, we are essentially treating the linear function as if it were a single  $x$ , then dividing the result by the coefficient of  $x$  in that linear function.

**(ii) Substitution Method**

The method to be discussed here will eventually be applied to functions other than functions of a linear function; but the latter serve as a useful way of introducing the technique of “**Integration by Substitution**”.

In the integral of the form  $\int f(ax+b)dx$ , we may substitute  $u = ax+b$  proceeding as follows:

Suppose

$$z = \int f(ax + b)dx.$$

Then,

$$\frac{dz}{dx} = f(ax + b).$$

That is,

$$\frac{dz}{dx} = f(u).$$

But,

$$\frac{dz}{du} = \frac{dz}{dx} \cdot \frac{dx}{du} = f(u) \cdot \frac{dx}{du}.$$

Hence,

$$z = \int f(u) \frac{dx}{du} du.$$

**Note:**

The secret of this integration by substitution formula is that, apart from putting  $u = ax + b$  into  $f(ax + b)$ , we replace the symbol  $dx$  with  $\frac{dx}{du} \cdot du$ ; almost as if we had divided  $dx$  by  $du$  then immediately multiplied by it again, though, strictly, this would not be allowed since  $dx$  and  $du$  are not numbers.

**EXAMPLES**

1. Determine the indefinite integral

$$z = \int (2x + 3)^{12} dx.$$

**Solution**

Putting  $u = 2x + 3$  gives  $\frac{du}{dx} = 2$  and, hence,  $\frac{dx}{du} = \frac{1}{2}$ .

Thus,

$$z = \int u^{12} \cdot \frac{1}{2} du = \frac{u^{13}}{13} \times \frac{1}{2} + C.$$

That is,

$$z = \frac{(2x + 3)^{13}}{26} + C,$$

as before.

2. Determine the indefinite integral

$$z = \int \cos(3 - 5x)dx.$$

**Solution**

Putting  $u = 3 - 5x$  gives  $\frac{du}{dx} = -5$  and hence  $\frac{dx}{du} = -\frac{1}{5}$ .

Thus,

$$z = \int \cos u \cdot -\frac{1}{5}du = -\frac{1}{5} \sin u + C.$$

That is,

$$z = -\frac{1}{5} \sin(3 - 5x) + C,$$

as before.

3. Determine the indefinite integral

$$z = \int e^{4x+1}dx.$$

**Solution**

Putting  $u = 4x + 1$  gives  $\frac{du}{dx} = 4$  and, hence,  $\frac{dx}{du} = \frac{1}{4}$ .

Thus,

$$z = \int e^u \cdot \frac{1}{4}du = \frac{e^u}{4} + C.$$

That is,

$$z = \frac{e^{4x+1}}{4} + C,$$

as before.

4. Determine the indefinite integral

$$z = \int \frac{1}{7x + 3}dx.$$

**Solution**

Putting  $u = 7x + 3$  gives  $\frac{du}{dx} = 7$  and, hence,  $\frac{dx}{du} = \frac{1}{7}$ .

Thus,

$$z = \int \frac{1}{u} \cdot \frac{1}{7}du = \frac{1}{7} \ln u + C.$$

That is,

$$z = \frac{1}{7} \ln(7x + 3) + C,$$

as before.

### 12.1.3 EXERCISES

1. Integrate the following functions with respect to  $x$ :

(a)

$$x^5;$$

(b)

$$x^{\frac{3}{2}};$$

(c)

$$\frac{1}{x^6};$$

(d)

$$2x^2 - x + 5;$$

(e)

$$x^3 - 7x^2 + x + 1.$$

2. Use a substitution of the form  $u = ax + b$  in order to determine the following integrals:

(a)

$$\int \sin(5x - 6)dx;$$

(b)

$$\int e^{2x+11}dx;$$

(c)

$$\int (3x + 2)^6 dx.$$

3. Write down, by inspection, the indefinite integrals with respect to  $x$  of the following functions:

(a)

$$(1 + 2x)^{10};$$

(b)

$$e^{12x+4};$$

(c)

$$\frac{1}{3x-1};$$

(d)

$$\sin(3-5x);$$

(e)

$$\frac{9}{(4-x)^5};$$

(f)

$$\operatorname{cosec}^2(7x+1).$$

#### 12.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{x^6}{6} + C;$$

(b)

$$\frac{2}{5}x^{\frac{5}{2}} + C;$$

(c)

$$-\frac{1}{5x^5} + C;$$

(d)

$$\frac{2}{3}x^3 - \frac{1}{2}x^2 + 5x + C;$$

(e)

$$\frac{1}{4}x^4 - \frac{7}{3}x^3 + \frac{1}{2}x^2 + x + C.$$

2. (a)

$$-\frac{1}{5} \cos(5x - 6) + C;$$

(b)

$$\frac{1}{2} e^{2x+11} + C;$$

(c)

$$\frac{1}{21} (3x + 2)^7 + C.$$

3. (a)

$$\frac{1}{22} (1 + 2x)^{11} + C;$$

(b)

$$\frac{1}{12} e^{12x+4} + C;$$

(c)

$$\frac{1}{3} \ln(3x - 1) + C;$$

(d)

$$\frac{1}{5} \cos(3 - 5x) + C;$$

(e)

$$\frac{9}{4(4 - x)^4} + C;$$

(f)

$$-\frac{1}{7} \cot(7x + 1) + C.$$

## APPENDIX - A Table of Standard Integrals

$f(x)$	$\int f(x) dx$
$a$ (const.)	$ax$
$x^n$	$x^{n+1}/(n+1) \quad n \neq -1$
$1/x$	$\ln x$
$\sin ax$	$-(1/a) \cos ax$
$\cos ax$	$(1/a) \sin ax$
$\sec^2 ax$	$(1/a) \tan ax$
$\operatorname{cosec}^2 ax$	$-(1/a) \cot ax$
$\sec ax \cdot \tan ax$	$(1/a) \sec ax$
$\operatorname{cosec} ax \cdot \cot ax$	$-(1/a) \operatorname{cosec} ax$
$e^{ax}$	$(1/a)e^{ax}$
$a^x$	$a^x / \ln a$
$\sinh ax$	$(1/a) \cosh ax$
$\cosh ax$	$(1/a) \sinh ax$
$\operatorname{sech}^2 ax$	$(1/a) \tanh ax$
$\operatorname{sech} ax \cdot \tanh ax$	$-(1/a) \operatorname{sech} ax$
$\operatorname{cosech} ax \cdot \operatorname{coth} ax$	$-(1/a) \operatorname{cosech} ax$
$\cot ax$	$(1/a) \ln(\sin ax)$
$\tan ax$	$-(1/a) \ln(\cos ax)$
$\tanh ax$	$(1/a) \ln(\cosh ax)$
$\operatorname{coth} ax$	$(1/a) \ln(\sinh ax)$
$1/\sqrt{a^2 - x^2}$	$\sin^{-1}(x/a)$
$1/(a^2 + x^2)$	$(1/a) \tan^{-1}(x/a)$
$1/\sqrt{x^2 + a^2}$	$\sinh^{-1}(x/a)$ or $\ln(x + \sqrt{x^2 + a^2})$
$1/\sqrt{x^2 - a^2}$	$\cosh^{-1}(x/a)$ or $\ln(x + \sqrt{x^2 - a^2})$
$1/(a^2 - x^2)$	$(1/a) \tanh^{-1}(x/a)$ or $\frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right)$ when $ x  < a$ , $\frac{1}{2a} \ln\left(\frac{x+a}{x-a}\right)$ when $ x  > a$ .