

“JUST THE MATHS”

UNIT NUMBER

11.5

DIFFERENTIATION APPLICATIONS 5
(Maclaurin’s and Taylor’s series)

by

A.J.Hobson

11.5.1 Maclaurin’s series
11.5.2 Standard series
11.5.3 Taylor’s series
11.5.4 Exercises
11.5.5 Answers to exercises

UNIT 11.5 - DIFFERENTIATION APPLICATIONS 5

MACLAURIN'S AND TAYLOR'S SERIES

11.5.1 MACLAURIN'S SERIES

One of the simplest kinds of function to deal with, in either algebra or calculus, is a polynomial (see Unit 1.8). Polynomials are easy to substitute numerical values into and they are easy to differentiate.

One useful application of the present section is to approximate, to a polynomial, functions which are not already in polynomial form.

THE GENERAL THEORY

Suppose $f(x)$ is a given function of x which is not in the form of a polynomial, and let us assume that it may be expressed in the form of an infinite series of ascending powers of x ; that is, a “**power series**”, (see Unit 2.4).

More specifically, we assume that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

This assumption cannot be justified unless there is a way of determining the “**coefficients**”, a_0, a_1, a_2, a_3, a_4 , etc.; but this is possible as an application of differentiation as we now show:

(a) Firstly, if we substitute $x = 0$ into the assumed formula for $f(x)$, we obtain $f(0) = a_0$; in other words,

$$a_0 = f(0).$$

(b) Secondly, if we differentiate the assumed formula for $f(x)$ once with respect to x , we obtain

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

which, on substituting $x = 0$, gives $f'(0) = a_1$; in other words,

$$a_1 = f'(0).$$

(c) Differentiating a second time leads to the result that

$$f''(x) = 2a_2 + (3 \times 2)a_3x + (4 \times 3)a_4x^2 + \dots$$

which, on substituting $x = 0$ gives $f''(0) = 2a_2$; in other words,

$$a_2 = \frac{1}{2}f''(0).$$

(d) Differentiating yet again leads to the result that

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4x + \dots$$

which, on substituting $x = 0$ gives $f'''(0) = (3 \times 2)a_3$; in other words,

$$a_3 = \frac{1}{3!}f'''(0).$$

(e) Continuing this process with further differentiation will lead to the general formula

$$a_n = \frac{1}{n!}f^{(n)}(0),$$

where $f^{(n)}(0)$ means the value, at $x = 0$ of the n -th derivative of $f(x)$.

Summary

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

This is called the “**Maclaurin’s series for $f(x)$** ”.

Notes:

(i) We must assume, ofcourse, that all of the derivatives of $f(x)$ exist at $x = 0$ in the first place; otherwise the above result is invalid.

It is also necessary to examine, for convergence or divergence, the Maclaurin’s series obtained

for a particular function. The result may not be used when the series diverges; (see Units 2.3 and 2.4).

(b) If x is small and it is possible to neglect powers of x after the n -th power, then Maclaurin's series approximates $f(x)$ to a polynomial of degree n .

11.5.2 STANDARD SERIES

Here, we determine the Maclaurin's series for some of the functions which occur frequently in the applications of mathematics to science and engineering. The ranges of values of x for which the results are valid will be stated without proof.

1. The Exponential Series

- | | |
|---------------------------|----------------------------------|
| (i) $f(x) \equiv e^x$; | hence, $f(0) = e^0 = 1$. |
| (ii) $f'(x) = e^x$; | hence, $f'(0) = e^0 = 1$. |
| (iii) $f''(x) = e^x$; | hence, $f''(0) = e^0 = 1$. |
| (iv) $f'''(x) = e^x$; | hence, $f'''(0) = e^0 = 1$. |
| (v) $f^{(iv)}(x) = e^x$; | hence, $f^{(iv)}(0) = e^0 = 1$. |

Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

and it may be shown that this series is valid for all values of x .

2. The Sine Series

- | | |
|------------------------------|-------------------------------------|
| (i) $f(x) \equiv \sin x$; | hence, $f(0) = \sin 0 = 0$. |
| (ii) $f'(x) = \cos x$; | hence, $f'(0) = \cos 0 = 1$. |
| (iii) $f''(x) = -\sin x$; | hence, $f''(0) = -\sin 0 = 0$. |
| (iv) $f'''(x) = -\cos x$; | hence, $f'''(0) = -\cos 0 = -1$. |
| (v) $f^{(iv)}(x) = \sin x$; | hence, $f^{(iv)}(0) = \sin 0 = 0$. |
| (vi) $f^{(v)}(x) = \cos x$; | hence, $f^{(v)}(0) = \cos 0 = 1$. |

Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

and it may be shown that this series is valid for all values of x .

(b) When n is a negative integer or a fraction, the expansion of $(1+x)^n$ in ascending powers of x is an **infinite** series.

Here, we examine the proof of the Binomial Formula.

(i) $f(x) \equiv (1+x)^n$; hence, $f(0) = 1$.

(ii) $f'(x) = n(1+x)^{n-1}$; hence, $f'(0) = n$.

(iii) $f''(x) = n(n-1)(1+x)^{n-2}$; hence, $f''(0) = n(n-1)$.

(iv) $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$; hence, $f'''(0) = n(n-1)(n-2)$.

(v) $f^{(iv)}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$; hence, $f^{(iv)}(0) = n(n-1)(n-2)(n-3)$.

Thus,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \dots$$

If n is a positive integer, all of the derivatives of $(1+x)^n$ after the n -th derivative are identically equal to zero; so the series is a finite series ending with the term in x^n .

In all other cases, the series is an infinite series and it may be shown that it is valid whenever $-1 < x \leq 1$.

EXAMPLES

1. Use the Maclaurin's series for $\sin x$ to evaluate

$$\lim_{x \rightarrow 0} \frac{x + \sin x}{x(x+1)}.$$

Solution

Substituting the series for $\sin x$ gives

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^2 + x} \\ = \lim_{x \rightarrow 0} \frac{2x - \frac{x^3}{6} + \frac{x^5}{120} - \dots}{x^2 + x} \\ = \lim_{x \rightarrow 0} \frac{2 - \frac{x^2}{6} + \frac{x^4}{120} - \dots}{x + 1} = 2. \end{aligned}$$

2. Use a Maclaurin's series to evaluate $\sqrt{1.01}$ correct to six places of decimals.

Solution

We shall consider the expansion of the function $(1+x)^{\frac{1}{2}}$ and then substitute $x = 0.01$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \dots$$

That is,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

Substituting $x = 0.01$ gives

$$\begin{aligned} \sqrt{1.01} &= 1 + \frac{1}{2} \times 0.01 - \frac{1}{8} \times 0.0001 + \frac{1}{16} \times 0.000001 - \dots \\ &= 1 + 0.005 - 0.0000125 + 0.0000000625 - \dots \end{aligned}$$

The fourth term will not affect the sixth decimal place in the result given by the first three terms; and this is equal to 1.004988 correct to six places of decimals.

3. Assuming the Maclaurin's series for e^x and $\sin x$ and assuming that they may be multiplied together term-by-term, obtain the expansion of $e^x \sin x$ in ascending powers of x as far as the term in x^5 .

Solution

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{120} + \dots\right) \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} + \dots \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \dots \end{aligned}$$

11.5.3 TAYLOR'S SERIES

A useful consequence of Maclaurin's series is known as Taylor's series and one form of it may be stated as follows:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \dots$$

Proof:

To obtain this result from Maclaurin's series, we simply let $f(x+h) \equiv F(x)$. Then,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0) + \dots$$

But, $F(0) = f(h)$, $F'(0) = f'(h)$, $F''(0) = f''(h)$, $F'''(0) = f'''(h)$, . . . which proves the result.

Note: An alternative form of Taylor's series, often used for approximations, may be obtained by interchanging the symbols x and h to give

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

EXAMPLE

Given that $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use Taylor's series to evaluate $\sin(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{4}$ and $h = 0.01$.

Solution

Using the sequence of derivatives as in the Maclaurin's series for $\sin x$, we have

$$\sin(x+h) = \sin x + h \cos x - \frac{h^2}{2!} \sin x - \frac{h^3}{3!} \cos x + \dots$$

Substituting $x = \frac{\pi}{4}$ and $h = 0.01$, we obtain

$$\begin{aligned} \sin\left(\frac{\pi}{4} + 0.01\right) &= \frac{1}{\sqrt{2}} \left(1 + 0.01 - \frac{(0.01)^2}{2!} - \frac{(0.01)^3}{3!} + \dots\right) \\ &= \frac{1}{\sqrt{2}}(1 + 0.01 - 0.00005 - 0.000000017 + \dots) \end{aligned}$$

The fourth term does not affect the fifth decimal place in the sum of the first three terms; and so

$$\sin\left(\frac{\pi}{4} + 0.01\right) \simeq \frac{1}{\sqrt{2}} \times 1.00995 \simeq 0.71414$$

11.5.4 EXERCISES

1. Determine the first three non-vanishing terms of the Maclaurin's series for the function $\sec x$.
2. Determine the Maclaurin's series for the function $\tan x$ as far as the term in x^5 .
3. Determine the Maclaurin's series for the function $\ln(1 + e^x)$ as far as the term in x^4 .
4. Use the Maclaurin's series for the function e^x to deduce the expansion, in ascending powers of x of the function e^{-x} and then use these two series to obtain the expansion, in ascending powers of x , of the functions

(a)

$$\frac{e^x + e^{-x}}{2} (\equiv \cosh x);$$

(b)

$$\frac{e^x - e^{-x}}{2} (\equiv \sinh x).$$

5. Use the Maclaurin's series for the function $\cos x$ and the Binomial Series for the function $\frac{1}{1+x}$ to obtain the expansion of the function

$$\frac{\cos x}{1+x}$$

in ascending powers of x as far as the term in x^4 .

6. From the Maclaurin's series for the function $\cos x$, deduce the expansions of the functions $\cos 2x$ and $\sin^2 x$ as far as the term in x^4 .

7. Use appropriate Maclaurin's series to evaluate the following limits:

(a)

$$\lim_{x \rightarrow 0} \left[\frac{e^x + e^{-x} - 2}{2 \cos 2x - 2} \right];$$

(b)

$$\lim_{x \rightarrow 0} \left[\frac{\sin^2 x - x^2 \cos x}{x^4} \right].$$

8. Use a Maclaurin's series to evaluate $\sqrt[3]{1.05}$ correct to four places of decimals.

9. Expand $\cos(x + h)$ as a series of ascending powers of h .

Given that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, evaluate $\cos(x + h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{6}$ and $h = -0.05$.

11.5.5 ANSWERS TO EXERCISES

1.

$$1 + \frac{x^2}{2} + \frac{x^4}{8} + \dots$$

2.

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

3.

$$\ln 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$$

4. (a)

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots;$$

(b)

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

5.

$$1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13x^4}{24} - \dots$$

6.

$$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} - \dots$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \dots$$

7. (a) $-\frac{1}{4}$, (b) $\frac{1}{6}$

8. 1.0164

9. 0.74156