

“JUST THE MATHS”

UNIT NUMBER

10.8

**DIFFERENTIATION 8
(Higher derivatives)**

by

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10.8.1 The theory

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UNIT 10.8 - DIFFERENTIATION 8

HIGHER DERIVATIVES

10.8.1 THE THEORY

In most of the examples (seen in earlier Units) on differentiating a function of x with respect to x , the result obtained has been **another** function of x . In general, this **will** be the case and the possibility arises of differentiating again with respect to x .

This would occur, for example, in the case when the formula

$$y = f(x)$$

represents the distance, y , travelled by a moving object at time, x .

The **speed** of the moving object is the rate of increase of distance with respect to time; that is, $\frac{dy}{dx}$. But a second quantity called **acceleration** is defined as the rate of increase of speed with respect to time. It is therefore represented by the symbol

$$\frac{d}{dx} \left[\frac{dy}{dx} \right];$$

but this is usually written as

$$\frac{d^2y}{dx^2}$$

and is pronounced “d two y by dx squared”.

We could, if necessary, differentiate over and over again to obtain the derivatives of order three, four, etc., namely

$$\frac{d^3y}{dx^3} \text{ and } \frac{d^4y}{dx^4}, \text{ etc.}$$

EXAMPLES

1. If $y = \sin 2x$, show that

$$\frac{d^2y}{dx^2} + 4y = 0.$$

Solution

Firstly,

$$\frac{dy}{dx} = 2 \cos 2x,$$

so that, on differentiating a second time, we obtain

$$\frac{d^2y}{dx^2} = -4 \sin 2x = -4y.$$

Hence, the result follows.

2. If $y = x^4$, show that every derivative of y with respect to x after the fourth derivative is zero.

Solution

$$\frac{dy}{dx} = 4x^3;$$

$$\frac{d^2y}{dx^2} = 12x^2;$$

$$\frac{d^3y}{dx^3} = 24x;$$

$$\frac{d^4y}{dx^4} = 24.$$

We now have a constant function, so that all future derivatives will be zero.

Note:

In general, every derivative of $y = x^n$ after the n -th derivative will be zero.

3. If $x = 3t^2$ and $y = 6t$, obtain an expression for $\frac{d^2y}{dx^2}$ in terms of t .

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

giving

$$\frac{dy}{dx} = \frac{6}{6t} = \frac{1}{t}.$$

In order to differentiate again with respect to x , we observe that, in the formula for the first derivative with respect to x , we need to replace y with $\frac{dy}{dx}$.

That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d \left[\frac{dy}{dx} \right]}{dx}.$$

Hence,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left[\frac{dy}{dx} \right]}{\frac{dx}{dt}}.$$

In the present example, therefore, we obtain

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left[\frac{1}{t} \right]}{6t} = \frac{-\frac{1}{t^2}}{6t} = -\frac{1}{6t^3}.$$

Note:

For a function $f(x)$, an alternative notation for the derivatives of order two, three, four, etc. is

$$f''(x), f'''(x), f^{(iv)}(x), \text{ etc.}$$

10.8.2 EXERCISES

1. Obtain expressions for $\frac{d^2y}{dx^2}$ in the following cases:

(a)

$$y = 4x^3 - 7x^2 + 5x - 17;$$

(b)

$$y = (3x - 2)^{10};$$

(c)

$$y = x^2 e^{4x};$$

(d)

$$y = \frac{x - 1}{x + 1}.$$

2. If $y = \sin 3x$, evaluate $\frac{d^2y}{dx^2}$ when $x = \frac{\pi}{4}$.

3. If $x^2 + y^2 - 2x + 2y = 23$ determine the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point where $x = -2$ and $y = 3$.

4. If $x = 3(1 - \cos \theta)$ and $y = 3(\theta - \sin \theta)$, show that

(a)

$$\frac{dy}{dx} = \tan \frac{\theta}{2};$$

(b)

$$\frac{d^2y}{dx^2} = \frac{1}{12 \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2}}.$$

5. If $y = 3e^{2x} \cos(2x - 3)$, verify that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0.$$

10.8.3 ANSWERS TO EXERCISES

1. (a)

$$\frac{d^2y}{dx^2} = 12x^2 - 14;$$

(b)

$$\frac{d^2y}{dx^2} = 810(3x - 2)^8;$$

(c)

$$\frac{d^2y}{dx^2} = e^{4x} [16x^2 + 16x + 2];$$

(d)

$$\frac{d^2y}{dx^2} = -\frac{4}{(x+1)^3}.$$

2.

$$-\frac{9}{\sqrt{2}}.$$

3.

$$\frac{3}{4} \text{ and } -\frac{25}{64}.$$