"JUST THE MATHS"

UNIT NUMBER

10.5

DIFFERENTIATION 5
(Implicit and parametric functions)

by

A.J.Hobson

UNIT 10.5 - DIFFERENTIATION 5
IMPLICIT AND PARAMETRIC FUNCTIONS

10.5.1 IMPLICIT FUNCTIONS

Some relationships between two variables x and y do not give y explicitly in terms of x (or x explicitly in terms of y); but, nevertheless, it is **implied** that one of the two variables is a function of the other. In the work which follows, we shall normally assume that y is a function of x.

Consider, for instance, the relationship

$$x^2 + y^2 = 16,$$

which is not explicit for either x or y but could, if desired, be written in one of the two forms

$$y = \pm\sqrt{16 - x^2} \quad \text{or} \quad x = \pm\sqrt{16 - y^2}.$$

By contrast, consider the relationship

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

In this case, there is no apparent way of stating either variable explicitly in terms of the other; yet we may still wish to calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ or even $\frac{\mathrm{d}x}{\mathrm{d}y}$.

Such relationships between x and y are said to be **"implicit relationships"** and, in the technique of **"implicit differentiation"**, we simply differentiate each term in the relationship with respect to the same variable without attempting to rearrange the formula.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + y^2 = 16.$$

Solution

Treating y^2 as a function of a function, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{2y} = -\frac{x}{y}.$$

It is perfectly acceptable that the result is expressed in terms of both x and y; this will normally happen.

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + 2xy^3 + y^5 = 4.$$

Solution

Treating y^3 and y^5 as functions of a function and using the Product Rule in the second term on the left hand side,

$$2x + 2\left[x.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.1\right] + 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

On rearrangement,

$$\left[6xy^2 + 5y^4\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -(2x + 2y^3).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + 2y^3}{6xy^2 + 5y^4}.$$

3. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

Solution

Differentiating throughout with respect to x and using both the Product Rule and the Function of a Function Rule, we obtain

$$x^2.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.2x + 9\cos(5x - 7y).\left[5 - 7\frac{\mathrm{d}y}{\mathrm{d}x}\right] = 0.$$

On rearrangment,

$$\left[3x^2y^2 - 63\cos(5x - 7y)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[2xy^3 + 45\cos(5x - 7y)\right].$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^3 + 45\cos(5x - 7y)}{3x^2y^2 - 63\cos(5x - 7y)}.$$

10.5.2 PARAMETRIC FUNCTIONS

In the geometry of straight lines, circles etc, we encounter **"parametric equations"** in which the variables x and y, related to each other by a formula, may each be expressed individually in terms of a third variable, usually t or θ, called a **"parameter"**.

In general, we write

$$x = x(t) \quad \text{and} \quad y = y(t);$$

but, in theory, we can imagine that t could be expressed explicitly in terms of x; so, essentially, y is a function of t, where t is a function of x. Hence, from the Function of a Function Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x}.$$

However, we are **not** given t explicitly in terms of x and it may not be practical to obtain it in this form. Therefore, we write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}.$$

This is the standard formula for differentiating y with respect to x from a pair of parametric equations.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t in the case when

$$x = 3t^2 \quad \text{and} \quad y = 6t.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6 \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 6t.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of θ in the case when

$$x = \sin^3\theta \quad \text{and} \quad y = \cos^3\theta.$$

Solution

$$\frac{dx}{d\theta} = 3\sin^2\theta.\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = -3\cos^2\theta.\sin\theta.$$

Hence,

$$\frac{dy}{dx} = \frac{-3\cos^2\theta.\sin\theta}{3\sin^2\theta.\cos\theta}.$$

That is,

$$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$

10.5.3 EXERCISES

1. Determine an expression for $\frac{dy}{dx}$ in the following cases:

   (a)
   $$x^2 + y^2 = 10x;$$

   (b)
   $$x^3 + y^3 - 3xy^2 = 8;$$

   (c)
   $$x^4 + 2x^2y^2 + y^4 = x;$$

   (d)
   $$xe^y = \cos y.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of the appropriate parameter in the following cases:

   (a)
   $$x = 3\sin\theta \quad \text{and} \quad y = 4\cos\theta;$$

   (b)
   $$x = 4t \quad \text{and} \quad y = \frac{4}{t};$$

(c)

$$x = (1-t)^{\frac{1}{2}} \quad \text{and} \quad y = (1-t^2)^{\frac{1}{2}}.$$

10.5.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5-x}{y};$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-x^2}{y^2-2xy};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-4x^3-4xy^2}{4\left(x^2y+y^3\right)};$$

(d)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^y}{xe^y+\sin y}.$$

2. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{3}\tan\theta;$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t}{(1+t)^{\frac{1}{2}}}.$$