

**“JUST THE MATHS”**

**UNIT NUMBER**

**10.3**

**DIFFERENTIATION 3**  
**(Elementary techniques of differentiation)**

by

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**10.3.1 Standard derivatives**  
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## UNIT 10.3 - DIFFERENTIATION 3

### ELEMENTARY TECHNIQUES OF DIFFERENTIATION

#### 10.3.1 STANDARD DERIVATIVES

In Unit 10.2, reference was made to the use of a table of standard derivatives and such a table can be found in the appendix at the end of this Unit.

However, for the time being, a very short list of standard derivatives is all that is necessary since other derivatives may be developed from them using techniques to be discussed later in this and subsequent Units.

$f(x)$	$f'(x)$
$a$ const.	0
$x^n$	$nx^{n-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$

**Note:**

In the work which now follows, standard derivatives may be used which have not, here, been obtained from first principles; but the student is expected to be able to quote results from a table of derivatives including those for which no proof has been given.

#### 10.3.2 RULES OF DIFFERENTIATION

**(a) Linearity**

Suppose  $f(x)$  and  $g(x)$  are two functions of  $x$  while  $A$  and  $B$  are constants. Then

$$\frac{d}{dx} [Af(x) + Bg(x)] = A\frac{d}{dx}[f(x)] + B\frac{d}{dx}[g(x)].$$

**Proof:**

The left-hand side is equivalent to

$$\begin{aligned} & \lim_{\delta x \rightarrow 0} \frac{[Af(x + \delta x) + Bg(x + \delta x)] - [Af(x) + Bg(x)]}{\delta x} \\ &= A \left[ \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \right] + B \left[ \lim_{\delta x \rightarrow 0} \frac{g(x + \delta x) - g(x)}{\delta x} \right] \end{aligned}$$

$$= A \frac{d}{dx}[f(x)] + B \frac{d}{dx}[g(x)].$$

The result, so far, deals with a “**linear combination**” of **two** functions of  $x$  but is easily extended to linear combinations of **three or more** functions of  $x$ .

### EXAMPLES

1. Write down the expression for  $\frac{dy}{dx}$  in the case when

$$y = 6x^2 + 2x^6 + 13x - 7.$$

#### Solution

Using the linearity property, the standard derivative of  $x^n$ , and the derivative of a constant, we obtain

$$\begin{aligned} \frac{dy}{dx} &= 6 \frac{d}{dx}[x^2] + 2 \frac{d}{dx}[x^6] + 13 \frac{d}{dx}[x^1] - \frac{d}{dx}[7] \\ &= 12x + 12x^5 + 13. \end{aligned}$$

2. Write down the derivative with respect to  $x$  of the function

$$\frac{5}{x^2} - 4 \sin x + 2 \ln x.$$

#### Solution

$$\begin{aligned} &\frac{d}{dx} \left[ \frac{5}{x^2} - 4 \sin x + 2 \ln x \right] \\ &= \frac{d}{dx} \left[ 5x^{-2} - 4 \sin x + 2 \ln x \right] \\ &= -10x^{-3} - 4 \cos x + \frac{2}{x} \\ &= \frac{-10}{x^3} - 4 \cos x + \frac{2}{x}. \end{aligned}$$

## (b) Composite Functions (or Functions of a Function)

### (i) Functions of a Linear Function

Expressions such as  $(5x + 2)^{16}$ ,  $\sin(2x + 3)$  and  $\ln(7 - 4x)$  may be called “**functions of a linear function**” and have the general form

$$f(ax + b),$$

where  $a$  and  $b$  are constants. The function  $f(x)$  would, of course, be the one obtained on replacing  $ax + b$  by a single  $x$ ; hence, in the above illustrations,  $f(x)$  would be  $x^{16}$ ,  $\sin x$  and  $\ln x$ , respectively.

Functions of a linear function may be differentiated as easily as  $f(x)$  itself on the strength of the following argument:

Suppose we write

$$y = f(u) \quad \text{where} \quad u = ax + b.$$

Suppose, also, that a small increase of  $\delta x$  in  $x$  gives rise to increases (positive or negative) of  $\delta y$  in  $y$  and  $\delta u$  in  $u$ . Then:

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta u} \frac{\delta u}{\delta x}.$$

Assuming that  $\delta y$  and  $\delta u$  tend to zero as  $\delta x$  tends to zero, we can say that

$$\frac{dy}{dx} = \lim_{\delta u \rightarrow 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x}.$$

That is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

This rule is called the “**Function of a Function Rule**” or “**Composite Function Rule**” or “**Chain Rule**” and has applications to a much wider class of composite functions than has so far been discussed. But, for the moment we restrict the discussion to functions of a linear function.

## EXAMPLES

1. Determine  $\frac{dy}{dx}$  when  $y = (5x + 2)^{16}$ .

**Solution**

First, we write  $y = u^{16}$  where  $u = 5x + 2$ .

Then,  $\frac{dy}{du} = 16u^{15}$  and  $\frac{du}{dx} = 5$ .

Hence,  $\frac{dy}{dx} = 16u^{15} \cdot 5 = 80(5x + 2)^{15}$ .

2. Determine  $\frac{dy}{dx}$  when  $y = \sin(2x + 3)$ .

**Solution**

First, we write  $y = \sin u$  where  $u = 2x + 3$ .

Then,  $\frac{dy}{du} = \cos u$  and  $\frac{du}{dx} = 2$ .

Hence,  $\frac{dy}{dx} = \cos u \cdot 2 = 2 \cos(2x + 3)$ .

3. Determine  $\frac{dy}{dx}$  when  $y = \ln(7 - 4x)$ .

**Solution**

First, we write  $y = \ln u$  where  $u = 7 - 4x$ .

Then,  $\frac{dy}{du} = \frac{1}{u}$  and  $\frac{du}{dx} = -4$ .

Hence,  $\frac{dy}{dx} = \frac{1}{u} \cdot (-4) = \frac{-4}{7-4x}$ .

**Note:**

It is hoped that the student will quickly appreciate how the fastest way to obtain the derivative of a function of a linear function is to treat the expression  $ax + b$  initially as if it were a single  $x$ ; then, multiply the final result by the constant value,  $a$ .

### (ii) Functions of a Function in general

The formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

is in no way dependent on the fact that the examples so far used to illustrate it have involved functions of a linear function. Exactly the same formula may be used for the composite function

$$f[g(x)],$$

whatever the functions  $f(x)$  and  $g(x)$  happen to be. All we need to do is to write

$$y = f(u) \quad \text{where} \quad u = g(x),$$

then apply the formula.

### EXAMPLES

1. Determine an expression for  $\frac{dy}{dx}$  in the case when

$$y = (x^2 + 7x - 3)^4.$$

#### Solution

Letting  $y = u^4$  where  $u = x^2 + 7x - 3$ , we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot (2x + 7) \\ &= 4(x^2 + 7x - 3)^3(2x + 7). \end{aligned}$$

2. Determine an expression for  $\frac{dy}{dx}$  in the case when

$$y = \ln(x^2 - 3x + 1).$$

#### Solution

Letting  $y = \ln u$  where  $u = x^2 - 3x + 1$ , we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (2x - 3) = \frac{2x - 3}{x^2 - 3x + 1}.$$

3. Determine the value of  $\frac{dy}{dx}$  at  $x = 1$  in the case when

$$y = 2 \sin(5x^2 - 1) + 19x.$$

#### Solution

Consider, first, the function  $2 \sin(5x^2 - 1)$  which we shall call  $z$ .

Its derivative is  $\frac{dz}{dx}$ , where  $z = 2 \sin(5x^2 - 1)$ .

Let  $z = 2 \sin u$  where  $u = 5x^2 - 1$ ; then,

$$\frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx} = 2 \cos u \cdot 10x = 20x \cos(5x^2 - 1).$$

Hence, the complete derivative is given by

$$\frac{dy}{dx} = 20x \cos(5x^2 - 1) + 19.$$

Finally, when  $x = 1$ , this derivative has the value  $20 \cos 4 + 19$ , which is approximately equal to 5.927, remembering that the calculator must be in **radian mode**.

**Note:**

Again, it is hoped that the student will appreciate how there is a faster way of differentiating composite functions in general. We simply treat  $g(x)$  initially as if it were a single  $x$ , then multiply by  $g'(x)$  afterwards.

For example,

$$\frac{d}{dx} [\sin^3 x] = \frac{d}{dx} [(\sin x)^3] = 3(\sin x)^2 \cdot \cos x = 3\sin^2 x \cdot \cos x.$$

### 10.3.3 EXERCISES

1. Determine an expression for  $\frac{dy}{dx}$  in the following cases:

(a)

$$y = 3x^3 - 8x^2 + 11x + 9;$$

(b)

$$y = 10 \cos x + 5 \sin x - 14x^7;$$

(c)

$$y = (2x - 7)^5;$$

(d)

$$y = (2 - 5x)^{-\frac{5}{2}};$$

(e)

$$y = \sin\left(\frac{\pi}{2} - x\right); \text{ that is, } \cos x;$$

(f)

$$y = \cos(4x + 1);$$

(g)

$$y = \ln(4 - 2x);$$

(h)

$$y = \ln\left[\frac{3x - 8}{6x + 2}\right].$$

2. Determine an expression for  $\frac{dy}{dx}$  in the cases when

(a)

$$y = (4 - 7x^3)^8;$$

(b)

$$y = (x^2 + 1)^{\frac{3}{2}};$$

(c)

$$y = \cos^5 x;$$

(d)

$$y = \ln(\ln x).$$

3. If  $y = \sin(\cos x)$ , evaluate  $\frac{dy}{dx}$  at  $x = \frac{\pi}{2}$ .

4. If  $y = \cos(7x^5 - 3)$ , evaluate  $\frac{dy}{dx}$  at  $x = 1$ .

### 10.3.4 ANSWERS TO EXERCISES

1. (a)

$$9x^2 - 16x + 11;$$

(b)

$$-10 \sin x + 5 \cos x - 98x^6;$$

(c)

$$10(2x - 7)^4;$$

(d)

$$\frac{25}{2}(2 - 5x)^{-\frac{7}{2}};$$

(e)

$$-\cos\left(\frac{\pi}{2} - x\right); \text{ that is, } -\sin x;$$

(f)

$$-4\sin(4x + 1);$$

(g)

$$\frac{-2}{4 - 2x} \text{ or } \frac{2}{2x - 4};$$

(h)

$$\frac{3}{3x - 8} - \frac{6}{6x + 2} = \frac{54}{(3x - 8)(6x + 2)}.$$

2. (a)

$$-168x^2(4 - 7x^3)^7;$$

(b)

$$3x(x^2 + 1)^{\frac{1}{2}};$$

(c)

$$-5\cos^4 x \cdot \sin x;$$

(d)

$$\frac{1}{x \ln x}.$$

3.

$$-1$$

4.

$$-35 \sin 4 \cong 26.488$$

## APPENDIX - A Table of Standard Derivatives

$f(x)$	$f'(x)$
$a$ (const.)	$0$
$x^n$	$nx^{n-1}$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
$\cot ax$	$-a \operatorname{cosec}^2 ax$
$\sec ax$	$a \sec ax \cdot \tan ax$
$\operatorname{cosec} ax$	$-a \operatorname{cosec} ax \cdot \cot ax$
$\ln x$	$1/x$
$e^{ax}$	$ae^{ax}$
$a^x$	$a^x \cdot \ln a$
$\sinh ax$	$a \cosh ax$
$\cosh ax$	$a \sinh ax$
$\tanh ax$	$a \operatorname{sech}^2 ax$
$\operatorname{sech} ax$	$-a \operatorname{sech} ax \cdot \tanh ax$
$\operatorname{cosech} ax$	$-a \operatorname{cosech} ax \cdot \coth x$
$\ln(\sin x)$	$\cot x$
$\ln(\cos x)$	$-\tan x$
$\ln(\sinh x)$	$\coth x$
$\ln(\cosh x)$	$\tanh x$
$\sin^{-1}(x/a)$	$1/\sqrt{a^2 - x^2}$
$\cos^{-1}(x/a)$	$-1/\sqrt{a^2 - x^2}$
$\tan^{-1}(x/a)$	$a/(a^2 + x^2)$
$\sinh^{-1}(x/a)$	$1/\sqrt{x^2 + a^2}$
$\cosh^{-1}(x/a)$	$1/\sqrt{x^2 - a^2}$
$\tanh^{-1}(x/a)$	$a/(a^2 - x^2)$