"JUST THE MATHS"

UNIT NUMBER

1.9

ALGEBRA 9
(The theory of partial fractions)

by

A.J.Hobson

1.9.1 Introduction
1.9.2 Standard types of partial fraction problem
1.9.3 Exercises
1.9.4 Answers to exercises

UNIT 1.9 - ALGEBRA 9 - THE THEORY OF PARTIAL FRACTIONS

1.9.1 INTRODUCTION

The theory of partial fractions applies chiefly to the ratio of two polynomials in which the degree of the numerator is strictly less than that of the denominator. Such a ratio is called a **"proper rational function"**.

For a rational function which is not proper, it is necessary first to use long division of polynomials in order to express it as the sum of a polynomial and a proper rational function.

RESULT

A proper rational function whose denominator has been factorised into its irreducible factors can be expressed as a sum of proper rational functions, called **"partial fractions"**; the denominators of the partial fractions are the irreducible factors of the denominator in the original rational function.

ILLUSTRATION

From previous work on fractions, it can be verified that

$$\frac{1}{2x+3} + \frac{3}{x-1} \equiv \frac{7x+8}{(2x+3)(x-1)}$$

and the expression on the left hand side may be interpretted as the decomposition into partial fractions of the expression on the right hand side.

1.9.2 STANDARD TYPES OF PARTIAL FRACTION PROBLEM

(a) Denominator of the given rational function has all linear factors.

EXAMPLE

Express the rational function

$$\frac{7x+8}{(2x+3)(x-1)}$$

in partial fractions.

Solution

There will be two partial fractions each of whose numerator must be of lower degree than 1; i.e. it must be a **constant**

We write

$$\frac{7x+8}{(2x+3)(x-1)} \equiv \frac{A}{2x+3} + \frac{B}{x-1}.$$

Multiplying throughout by $(2x+3)(x-1)$, we obtain

$$7x + 8 \equiv A(x-1) + B(2x+3).$$

In order to determine A and B, any two suitable values of x may be substituted on both sides; and the most obvious values in this case are $x = 1$ and $x = -\frac{3}{2}$.

It may, however, be argued that these two values of x must be disallowed since they cause denominators in the first identity above to take the value zero.

Nevertheless, we shall use these values in the second identity above since the arithmetic involved is negligibly different from taking values infinitesimally close to $x = 1$ and $x = -\frac{3}{2}$.

Substituting $x = 1$ gives

$$7 + 8 = B(2+3).$$

Hence,

$$B = \frac{7+8}{2+3} = \frac{15}{5} = 3.$$

Substituting $x = -\frac{3}{2}$ gives

$$7 \times -\frac{3}{2} + 8 = A(-\frac{3}{2} - 1).$$

Hence,

$$A = \frac{7 \times -\frac{3}{2} + 8}{-\frac{3}{2} - 1} = \frac{-\frac{5}{2}}{-\frac{5}{2}} = 1.$$

We conclude that

$$\frac{7x+8}{(2x+3)(x-1)} = \frac{1}{2x+3} + \frac{3}{x-1}.$$

The "Cover-up" Rule
A useful time-saver when the factors in the denominator of the given rational function are linear is to use the following routine which is equivalent to the method described above:

To obtain the constant numerator of the partial fraction for a particular linear factor, $ax+b$, in the original denominator, cover up $ax+b$ in the original rational function and then substitute $x = -\frac{b}{a}$ into what remains.

ILLUSTRATION
In the above example, we may simply cover up $x-1$, then substitute $x = 1$ into the fraction

$$\frac{7x+8}{2x+3}.$$

Then we may cover up $2x+3$ and substitute $x = -\frac{3}{2}$ into the fraction

$$\frac{7x+8}{x-1}.$$

Note:
We shall see later how the cover-up rule can also be brought into effective use when not all of the factors in the denominator of the given rational function are linear.

(b) Denominator of the given rational function contains one linear and one quadratic factor

EXAMPLE

Express the rational function

$$\frac{3x^2+9}{(x-5)(x^2+2x+7)}$$

in partial fractions.

Solution

We should observe firstly that the quadratic factor will not reduce conveniently into two linear factors. If it did, the method would be as in the previous paragraph. Hence we may write

$$\frac{3x^2+9}{(x-5)(x^2+2x+7)} \equiv \frac{A}{x-5} + \frac{Bx+C}{x^2+2x+7},$$

noticing that the second partial fraction may contain an x term in its numerator, yet still be a proper rational function.

Multiplying throughout by $(x-5)(x^2+2x+7)$, we obtain

$$3x^2+9 \equiv A(x^2+2x+7) + (Bx+C)(x-5).$$

A convenient value of x to substitute on both sides is $x = 5$ which gives

$$3 \times 5^2 + 9 = A(5^2 + 2 \times 5 + 7).$$

That is, $84 = 42A$ or $A = 2$.

No other convenient values of x may be substituted; but two polynomial expressions can be identical only if their corresponding coefficients are the same in value. We therefore equate suitable coefficients to find B and C; usually, the coefficients of the highest and lowest powers of x.

Equating coefficients of x^2, $3 = A + B$ and hence $B = 1$.

Equating constant terms (the coefficients of x^0), $9 = 7A - 5C = 14 - 5C$ and hence $C = 1$.

The result is therefore

$$\frac{3x^2 + 9}{(x-5)(x^2+2x+7)} \equiv \frac{2}{x-5} + \frac{x+1}{x^2+2x+7}.$$

Observations
It is easily verified that the value of A may be calculated by means of the cover-up rule, as in paragraph **(a)**; and, having found A, the values of B and C could be found by cross multiplying the numerators and denominators in the expression

$$\frac{2}{x-5} + \frac{?x+?}{x^2+2x+7}$$

in order to arrive at the numerator of the original rational function. This process essentially compares the coefficients of x^2 and x^0 as before.

(c) Denominator of the given rational function contains a repeated linear factor

In general, examples of this kind will not be more complicated than for a rational function with one repeated linear factor together with either a non-repeated linear factor or a quadratic factor.

EXAMPLE

Express the rational function

$$\frac{9}{(x+1)^2(x-2)}$$

in partial fractions.

Solution

First we observe that, from paragraph **(b)**, the partial fraction corresponding to the repeated linear factor would be of the form

$$\frac{Ax+B}{(x+1)^2};$$

but this may be written

$$\frac{A(x+1)+B-A}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{B-A}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2}.$$

Thus, a better form of statement for the problem as a whole, is

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-2}.$$

Eliminating fractions, we obtain

$$9 \equiv A(x+1)(x-2) + C(x-2) + D(x+1)^2.$$

Putting $x = -1$ gives $9 = -3C$ so that $C = -3$.

Putting $x = 2$ gives $9 = 9D$ so that $D = 1$.

Equating coefficients of x^2 gives $0 = A + D$ so that $A = -1$.

Therefore,

$$\frac{9}{(x+1)^2(x-2)} \equiv -\frac{1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}.$$

Notes:
(i) Similar partial fractions may be developed for higher repeated powers so that, for a repeated linear factor of power of n, there will be n corresponding partial fractions, each with a constant numerator. The labels for these numerators in future will be taken as A, B, C, etc. in sequence.

(ii) We observe that the numerator above the repeated factor itself (D in this case) could actually have been obtained by the cover-up rule; covering up $(x+1)^2$ in the original rational function, then substituting $x = -1$ into the rest.

(d) Keily's Method

A useful method for repeated linear factors is to use these factors one at a time, keeping the rest outside the expression as a factor.

EXAMPLE

Express the rational function

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{9}{(x+1)(x-2)}\right]$$

in partial fractions.

Solution

Using the cover-up rule inside the square brackets,

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{-3}{x+1} + \frac{3}{x-2}\right]$$

$$\equiv -\frac{3}{(x+1)^2} + \frac{3}{(x+1)(x-2)};$$

and, again by cover-up rule,

$$\equiv -\frac{3}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x-2}$$

as before.

Warning
Care must be taken with Keily's method when, even though the original rational function is proper, the resulting expression inside the square brackets is improper. This would have occurred, for instance, if the problem given had been

$$\frac{9x^2}{(x+1)^2(x-2)},$$

leading to

$$\frac{1}{x+1}\left[\frac{9x^2}{(x+1)(x-2)}\right].$$

In this case, long division would have to be used inside the square brackets before proceeding with Keily's method.

For such examples, it is probably better to use the method of paragraph **(c)**.

1.9.3 EXERCISES

Express the following rational functions in partial fractions:

1. $$\frac{3x+5}{(x+1)(x+2)}.$$

2.

$$\frac{17x+11}{(x+1)(x-2)(x+3)}.$$

3.

$$\frac{3x^2-8}{(x-1)(x^2+x-7)}.$$

4.

$$\frac{2x+1}{(x+2)^2(x-3)}.$$

5.

$$\frac{9+11x-x^2}{(x+1)^2(x+2)}.$$

6.

$$\frac{x^5}{(x+2)(x-4)}.$$

1.9.4 ANSWERS TO EXERCISES

1.

$$\frac{2}{x+1}+\frac{1}{x+2}.$$

2.

$$\frac{1}{x+1}+\frac{3}{x-2}-\frac{4}{x+3}.$$

3.

$$\frac{1}{x-1}+\frac{2x+1}{x^2+x-7}.$$

4.

$$\frac{3}{5(x+2)^2}-\frac{7}{25(x+2)}+\frac{7}{25(x-3)}.$$

5.

$$-\frac{3}{(x+1)^2}+\frac{16}{x+1}-\frac{17}{x+2}.$$

6.

$$x^3+2x^2+12x+40+\frac{16}{3(x+2)}+\frac{512}{3(x-4)}.$$