

**“JUST THE MATHS”**

**UNIT NUMBER**

**1.5**

**ALGEBRA 5**

**(Manipulation of algebraic expressions)**

**by**

**A.J.Hobson**

- 1.5.1 Simplification of expressions**
- 1.5.2 Factorisation**
- 1.5.3 Completing the square in a quadratic expression**
- 1.5.4 Algebraic Fractions**
- 1.5.5 Exercises**
- 1.5.6 Answers to exercises**

## UNIT 1.5 - ALGEBRA 5 MANIPULATION OF ALGEBRAIC EXPRESSIONS

### 1.5.1 SIMPLIFICATION OF EXPRESSIONS

An algebraic expression will, in general, contain a mixture of alphabetical symbols together with one or more numerical quantities; some of these symbols and numbers may be bracketted together.

Using the Language of Algebra and the Laws of Algebra discussed earlier, the method of simplification is to remove brackets and collect together any terms which have the same format

Some elementary illustrations are as follows:

1.  $a + a + a + 3 + b + b + b + 8 \equiv 3a + 4b + 11$ .
2.  $11p^2 + 5q^7 - 8p^2 + q^7 \equiv 3p^2 + 6q^7$ .
3.  $a(2a - b) + b(a + 5b) - a^2 - 4b^2 \equiv 2a^2 - ab + ba + 5b^2 - a^2 - 4b^2 \equiv a^2 + b^2$ .

More frequently, the expressions to be simplified will involve symbols which represent both the constants and variables encountered in scientific work. Typical examples in pure mathematics use symbols like  $x$ ,  $y$  and  $z$  to represent the variable quantities.

Further illustrations use this kind of notation and, for simplicity, we shall omit the full-stop type of multiplication sign between symbols.

1.  $x(2x + 5) + x^2(3 - x) \equiv 2x^2 + 5x + 3x^2 - x^3 \equiv 5x^2 + 5x - x^3$ .
2.  $x^{-1}(4x - x^2) - 6(1 - 3x) \equiv 4 - x - 6 + 18x \equiv 17x - 2$ .

We need also to consider the kind of expression which involves two or more brackets multiplied together; but the routine is just an extension of what has already been discussed.

For example consider the expression

$$(a + b)(c + d).$$

Taking the first bracket as a single item for the moment, the Distributive Law gives

$$(a + b)c + (a + b)d.$$

Using the Distributive Law a second time gives

$$ac + bc + ad + bd.$$

In other words, each of the two terms in the first bracket are multiplied by each of the two terms in the second bracket, giving four terms in all.

Again, we illustrate with examples:

### EXAMPLES

1.  $(x + 3)(x - 5) \equiv x^2 + 3x - 5x - 15 \equiv x^2 - 2x - 15.$
2.  $(x^3 - x)(x + 5) \equiv x^4 - x^2 + 5x^3 - 5x.$
3.  $(x + a)^2 \equiv (x + a)(x + a) \equiv x^2 + ax + ax + a^2 \equiv x^2 + 2ax + a^2.$
4.  $(x + a)(x - a) \equiv x^2 + ax - ax - a^2 \equiv x^2 - a^2.$

The last two illustrations above are significant for later work because they incorporate, respectively, the standard results for a “**perfect square**” and “**the difference between two squares**”.

### 1.5.2 FACTORISATION

#### Introduction

In an algebraic context, the word “**factor**” means the same as “**multiplier**”. Thus, to factorise an algebraic expression, is to write it as a product of separate multipliers or factors.

Some simple examples will serve to introduce the idea:

### EXAMPLES

1.

$$3x + 12 \equiv 3(x + 4).$$

2.

$$8x^2 - 12x \equiv x(8x - 12) \equiv 4x(2x - 3).$$

3.

$$5x^2 + 15x^3 \equiv x^2(5 + 15x) \equiv 5x^2(1 + 3x).$$

4.

$$6x + 3x^2 + 9xy \equiv x(6 + 3x + 9y) \equiv 3x(2 + x + 3y).$$

**Note:**

When none of the factors can be broken down into simpler factors, the original expression is said to have been factorised into “**irreducible factors**”.

**Factorisation of quadratic expressions**

A “**quadratic expression**” is an expression of the form

$$ax^2 + bx + c,$$

where, usually,  $a$ ,  $b$  and  $c$  are fixed numbers (constants) while  $x$  is a variable number. The numbers  $a$  and  $b$  are called, respectively, the “**coefficients**” of  $x^2$  and  $x$  while  $c$  is called the “**constant term**”; but, for brevity, we often say that the quadratic expression has coefficients  $a$ ,  $b$  and  $c$ .

**Note:**

It is important that the coefficient  $a$  does not have the value zero otherwise the expression is not quadratic but “**linear**”.

The method of factorisation is illustrated by examples:

(a) **When the coefficient of  $x^2$  is 1**

**EXAMPLES**

1.

$$x^2 + 5x + 6 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn.$$

This implies that  $5 = m + n$  and  $6 = mn$  which, by inspection gives  $m = 2$  and  $n = 3$ .  
Hence

$$x^2 + 5x + 6 \equiv (x + 2)(x + 3).$$

2.

$$x^2 + 4x - 21 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn.$$

This implies that  $4 = m + n$  and  $-21 = mn$  which, by inspection, gives  $m = -3$  and  $n = 7$ . Hence

$$x^2 + 4x - 21 \equiv (x - 3)(x + 7).$$

**Notes:**

(i) In general, for simple cases, it is better to try to carry out the factorisation entirely by inspection. This avoids the cumbersome use of  $m$  and  $n$  in the above two examples as follows:

$$x^2 + 2x - 8 \equiv (x+?)(x+?).$$

The two missing numbers must be such that their sum is 2 and their product is  $-8$ . The required values are therefore  $-2$  and 4. Hence

$$x^2 + 2x - 8 \equiv (x - 2)(x + 4).$$

(ii) It is necessary, when factorising a quadratic expression, to be aware that either a perfect square or the difference of two squares might be involved. In these cases, the factorisation is a little simpler. For instance:

$$x^2 + 10x + 25 \equiv (x + 5)^2$$

and

$$x^2 - 64 \equiv (x - 8)(x + 8).$$

(iii) Some quadratic expressions will not conveniently factorise at all. For example, in the expression

$$x^2 - 13x + 2,$$

we cannot find two whole numbers whose sum is  $-13$  while, at the same time, their product is 2.

**(b) When the coefficient of  $x^2$  is not 1**

Quadratic expressions of this kind are usually more difficult to factorise than those in the previous paragraph. We first need to determine the possible pairs of factors of the coefficient of  $x^2$  and the possible pairs of factors of the constant term; then we need to consider the possible combinations of these which provide the correct factors of the quadratic expression.

**EXAMPLES**

1. To factorise the expression

$$2x^2 + 11x + 12,$$

we observe that 2 is the product of 2 and 1 only, while 12 is the product of either 12 and 1, 6 and 2 or 4 and 3. All terms of the quadratic expression are positive and hence we may try  $(2x+1)(x+12)$ ,  $(2x+12)(x+1)$ ,  $(2x+6)(x+2)$ ,  $(2x+2)(x+6)$ ,  $(2x+4)(x+3)$  and  $(2x+3)(x+4)$ . Only the last of these is correct and so

$$2x^2 + 11x + 12 \equiv (2x + 3)(x + 4).$$

2. To factorise the expression

$$6x^2 + 7x - 3,$$

we observe that 6 is the product of either 6 and 1 or 3 and 2 while 3 is the product of 3 and 1 only. A negative constant term implies that, in the final result, its two factors must have opposite signs. Hence we may try  $(6x+3)(x-1)$ ,  $(6x-3)(x+1)$ ,  $(6x+1)(x-3)$ ,  $(6x-1)(x+3)$ ,  $(3x+3)(2x-1)$ ,  $(3x-3)(2x+1)$ ,  $(3x+1)(2x-3)$  and  $(3x-1)(2x+3)$ . Again, only the last of these is correct and so

$$6x^2 + 7x - 3 \equiv (3x - 1)(2x + 3).$$

**Note:**

The more factors there are in the coefficients considered, the more possibilities there are to try of the final factorisation.

### 1.5.3 COMPLETING THE SQUARE IN A QUADRATIC EXPRESSION

The following work is based on the standard expansions

$$(x + a)^2 \equiv x^2 + 2ax + a^2$$

and

$$(x - a)^2 \equiv x^2 - 2ax + a^2.$$

Both of these last expressions are called “**complete squares**” (or “**perfect squares**”) in which we observe that one half of the coefficient of  $x$ , when multiplied by itself, gives the constant term. That is

$$\left(\frac{1}{2} \times 2a\right)^2 = a^2.$$

### ILLUSTRATIONS

1.

$$x^2 + 6x + 9 \equiv (x + 3)^2.$$

2.

$$x^2 - 8x + 16 \equiv (x - 4)^2.$$

3.

$$4x^2 - 4x + 1 \equiv 4 \left[ x^2 - x + \frac{1}{4} \right] \equiv 4 \left( x - \frac{1}{2} \right)^2.$$

Of course it may happen that a given quadratic expression is NOT a complete square; but, by using one half of the coefficient of  $x$ , we may express it as the sum or difference of a complete square and a constant. This process is called “**completing the square**”, and the following examples illustrate it:

### EXAMPLES

1.

$$x^2 + 6x + 11 \equiv (x + 3)^2 + 2.$$

2.

$$x^2 - 8x + 7 \equiv (x - 4)^2 - 9.$$

3.

$$\begin{aligned} 4x^2 - 4x + 5 &\equiv 4 \left[ x^2 - x + \frac{5}{4} \right] \\ &\equiv 4 \left[ \left( x - \frac{1}{2} \right)^2 - \frac{1}{4} + \frac{5}{4} \right] \\ &\equiv 4 \left[ \left( x - \frac{1}{2} \right)^2 + 1 \right] \\ &\equiv 4 \left( x - \frac{1}{2} \right)^2 + 4. \end{aligned}$$

### 1.5.4 ALGEBRAIC FRACTIONS

We first recall the basic rules for combining fractions, namely

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

We also note that a single algebraic fraction may sometimes be simplified by the cancellation of common factors between the numerator and the denominator.

### EXAMPLES

1.

$$\frac{5}{25 + 15x} \equiv \frac{1}{5 + 3x}, \text{ assuming that } x \neq -\frac{5}{3}.$$

2.

$$\frac{4x}{3x^2 + x} \equiv \frac{4}{3x + 1}, \text{ assuming that } x \neq 0 \text{ or } -\frac{1}{3}.$$

3.

$$\frac{x + 2}{x^2 + 3x + 2} \equiv \frac{x + 2}{(x + 2)(x + 1)} \equiv \frac{1}{x + 1}, \text{ assuming that } x \neq -1 \text{ or } -2.$$

These elementary principles may now be used with more advanced combinations of algebraic fractions

### EXAMPLES

1. Simplify the expression

$$\frac{3x + 6}{x^2 + 3x + 2} \times \frac{x + 1}{2x + 8}.$$

#### Solution

Using factorisation where possible, together with the rule for multiplying fractions, we obtain

$$\frac{3(x + 2)(x + 1)}{2(x + 4)(x + 1)(x + 2)} \equiv \frac{3}{2(x + 4)},$$

assuming that  $x \neq -1, -2$  or  $-4$ .

2. Simplify the expression

$$\frac{3}{x + 2} \div \frac{x}{2x + 4}.$$

#### Solution

Using factorisation where possible together with the rule for dividing fractions, we obtain

$$\frac{3}{x + 2} \times \frac{2x + 4}{x} \equiv \frac{3}{x + 2} \times \frac{2(x + 2)}{x} \equiv \frac{6}{x},$$

assuming that  $x \neq 0$  or  $-2$ .

3. Express

$$\frac{4}{x+y} - \frac{3}{y}$$

as a single fraction.

**Solution**

From the basic rule for adding and subtracting fractions, we obtain

$$\frac{4y - 3(x+y)}{(x+y)y} \equiv \frac{y - 3x}{(x+y)y},$$

assuming that  $y \neq 0$  and  $x \neq -y$ .

4. Express

$$\frac{x}{x+1} + \frac{4-x^2}{x^2-x-2}$$

as a single fraction.

**Solution**

This example could be tackled in the same way as the previous one but it is worth noticing that  $x^2 - x - 2 \equiv (x+1)(x-2)$ . Consequently, it is worth putting both fractions over the simplest common denominator, namely  $(x+1)(x-2)$ . Hence we obtain, if  $x \neq 2$  or  $-1$ ,

$$\frac{x(x-2)}{(x+1)(x-2)} + \frac{4-x^2}{(x+1)(x-2)} \equiv \frac{x^2 - 2x + 4 - x^2}{(x+1)(x-2)} \equiv \frac{2(2-x)}{(x+1)(x-2)} \equiv -\frac{2}{x+1}.$$

### 1.5.5 EXERCISES

1. Write down in their simplest forms

(a)  $5a - 2b - 3a + 6b$ ; (b)  $11p + 5q - 2q + p$ .

2. Simplify the following expressions:

(a)  $3x^2 - 2x + 5 - x^2 + 7x - 2$ ; (b)  $x^3 + 5x^2 - 2x + 1 + x - x^2$ .

3. Expand and simplify the following expressions:

(a)  $x(x^2 - 3x) + x^2(4x + 7)$ ; (b)  $(2x - 1)(2x + 1) - x^2 + 5x$ ;

(c)  $(x + 3)(2x^2 - 5)$ ; (d)  $2(3x + 1)^2 + 5(x - 7)^2$ .

4. Factorise the following expressions:

(a)  $xy + 4x^2y$ ; (b)  $2abc - 6ab^2$ ;

(c)  $\pi r^2 + 2\pi rh$ ; (d)  $2xy^2z + 4x^2z$ .

5. Factorise the following quadratic expressions:

(a)  $x^2 + 8x + 12$ ; (b)  $x^2 + 11x + 18$ ; (c)  $x^2 + 13x - 30$ ;

(d)  $3x^2 + 11x + 6$ ; (e)  $4x^2 - 12x + 9$ ; (f)  $9x^2 - 64$ .

6. Complete the square in the following quadratic expressions:

(a)  $x^2 - 10x - 26$ ; (b)  $x^2 - 5x + 4$ ; (c)  $7x^2 - 2x + 1$ .

7. Simplify the following:

(a)  $\frac{x^2+4x+4}{x^2+5x+6}$ ; (b)  $\frac{x^2-1}{x^2+2x+1}$ ,

assuming the values of  $x$  to be such that no denominators are zero.

8. Express each of the following as a single fraction:

(a)  $\frac{3}{x} + \frac{4}{y}$ ; (b)  $\frac{4}{x} - \frac{6}{2x}$ ;

(c)  $\frac{1}{x+1} + \frac{1}{x+2}$ ; (d)  $\frac{5x}{x^2+5x+4} - \frac{3}{x+4}$ ,

assuming that the values of  $x$  and  $y$  are such that no denominators are zero.

### 1.5.6 ANSWERS TO EXERCISES

1. (a)  $2a + 4b$ ; (b)  $12p + 3q$ .

2. (a)  $2x^2 + 5x + 3$ ; (b)  $x^3 + 4x^2 - x + 1$ .

3. (a)  $5x^3 + 4x^2$ ; (b)  $3x^2 + 5x - 1$ .

(c)  $2x^3 + 6x^2 - 5x - 15$ ; (d)  $23x^2 - 58x + 247$ .

4. (a)  $xy(1 + 4x)$ ; (b)  $2ab(c - 3b)$ ;

(c)  $\pi r(r + 2h)$ ; (d)  $2xz(y^2 + 2x)$ .

5. (a)  $(x + 2)(x + 6)$ ; (b)  $(x + 2)(x + 9)$ ; (c)  $(x - 2)(x + 15)$ ;

(d)  $(3x + 2)(x + 3)$ ; (e)  $(2x - 3)^2$ ; (f)  $(3x - 8)(3x + 8)$ .

6. (a)  $(x - 5)^2 - 51$ ; (b)  $(x - \frac{5}{2})^2 - \frac{9}{4}$ ; (c)  $7[(x - \frac{1}{7})^2 + \frac{6}{49}]$ .

7. (a)  $\frac{x+2}{x+3}$ ; (b)  $\frac{x-1}{x+1}$ .

8. (a)  $\frac{3y+4x}{xy}$ ; (b)  $\frac{1}{x}$ ;

(c)  $\frac{2x+3}{(x+1)(x+2)}$ ; (d)  $\frac{2x-3}{x^2+5x+4}$ .