"JUST THE MATHS"

UNIT NUMBER

1.3

ALGEBRA 3
(Indices and radicals (or surds))

by

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UNIT 1.3 - ALGEBRA 3 - INDICES AND RADICALS (or Surds)

1.3.1 INDICES

(a) Positive Integer Indices

It was seen earlier that, for any number a, a^2 denotes $a.a$, a^3 denotes a.a.a, a^4 denotes $a.a.a.a$ and so on.

Suppose now that a and b are arbitrary numbers and that m and n are natural numbers (i.e. positive whole numbers)

Then the following rules are the basic Laws of Indices:

Law No. 1

$$a^m \times a^n = a^{m+n}$$

Law No. 2

$$a^m \div a^n = a^{m-n}$$

assuming, for the moment, that m is greater than n.

Note:
It is natural to use this rule to give a definition to a^0 which would otherwise be meaningless.

Clearly $\frac{a^m}{a^m} = 1$ but the present rule for indices suggests that $\frac{a^m}{a^m} = a^{m-m} = a^0$.
Hence, we **define** a^0 to be equal to 1.

Law No. 3

$$(a^m)^n = a^{mn}$$

$$a^m b^m = (ab)^m$$

EXAMPLE

Simplify the expression,

$$\frac{x^2y^3}{z} \div \frac{xy}{z^5}.$$

Solution

The expression becomes

$$\frac{x^2y^3}{z} \times \frac{z^5}{xy} = xy^2z^4.$$

(b) Negative Integer Indices

Law No. 4

$$a^{-1} = \frac{1}{a}$$

Note:
It has already been mentioned that a^{-1} means the same as $\frac{1}{a}$; and the logic behind this statement is to maintain the basic Laws of Indices for negative indices as well as positive ones.

For example $\frac{a^m}{a^{m+1}}$ is clearly the same as $\frac{1}{a}$ but, using Law No. 2 above, it could also be thought of as $a^{m-[m+1]} = a^{-1}$.

Law No. 5

$$a^{-n} = \frac{1}{a^n}$$

Note:
This time, we may observe that $\frac{a^m}{a^{m+n}}$ is clearly the same as $\frac{1}{a^n}$; but we could also use Law No. 2 to interpret it as $a^{m-[m+n]} = a^{-n}$

Law No. 6

$$a^{-\infty} = 0$$

Note:
Strictly speaking, no power of a number can ever be equal to zero, but Law No. 6 asserts that a very large negative power of a number a gives a very small value; the larger the negative power, the smaller will be the value.

EXAMPLE

Simplify the expression,

$$\frac{x^5y^2z^{-3}}{x^{-1}y^4z^5} \div \frac{z^2x^2}{y^{-1}}.$$

Solution

The expression becomes

$$x^5y^2z^{-3}xy^{-4}z^{-5}y^{-1}z^{-2}x^{-2} = x^4y^{-3}z^{-10}.$$

(c) Rational Indices

(i) Indices of the form $\frac{1}{n}$ where n is a natural number.

In order to preserve Law No. 3, we interpret $a^{\frac{1}{n}}$ to mean a number which gives the value a when it is raised to the power n. It is called an **"n-th Root of a"** and, sometimes there is more than one value.

ILLUSTRATION

$$81^{\frac{1}{4}} = \pm 3 \text{ but } (-27)^{\frac{1}{3}} = -3 \text{ only.}$$

(ii) Indices of the form $\frac{m}{n}$ where m and n are natural numbers with no common factor.

The expression $y^{\frac{m}{n}}$ may be interpreted in two ways as either $(y^m)^{\frac{1}{n}}$ or $(y^{\frac{1}{n}})^m$. It may be shown that both interpretations give the same result but, sometimes, the arithmetic is shorter with one rather than the other.

ILLUSTRATION

$$27^{\frac{2}{3}} = 3^2 = 9 \text{ or } 27^{\frac{2}{3}} = 729^{\frac{1}{3}} = 9.$$

Note:
It may be shown that all of the standard laws of indices may be used for fractional indices.

1.3.2 RADICALS (or Surds)

The symbol "$\sqrt{}$" is called a **"radical"** (or **"surd"**). It is used to indicate the positive or **"principal"** square root of a number. Thus $\sqrt{16} = 4$ and $\sqrt{25} = 5$.

The number under the radical is called the **"radicand"**.

Most of our work on radicals will deal with square roots, but we may have occasion to use other roots of a number. For instance the **principal n-th root** of a number a is denoted by $\sqrt[n]{a}$, and is a number x such that $x^n = a$. The number n is called the **index** of the radical but, of course, when $n = 2$ we usually leave the index out.

ILLUSTRATIONS

1. $\sqrt[3]{64} = 4$ since $4^3 = 64$.
2. $\sqrt[3]{-64} = -4$ since $(-4)^3 = -64$.
3. $\sqrt[4]{81} = 3$ since $3^4 = 81$.
4. $\sqrt[5]{32} = 2$ since $2^5 = 32$.
5. $\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$.

Note:
If the index of the radical is an odd number, then the radicand may be positive or negative; but if the index of the radical is an even number, then the radicand may not be negative since no even power of a negative number will ever give a negative result.

(a) Rules for Square Roots

In preparation for work which will follow in the next section, we list here the standard rules for square roots:

(i) $(\sqrt{a})^2 = a$

(ii) $\sqrt{a^2} = |a|$

(iii) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(iv) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

assuming that all of the radicals can be evaluated.

ILLUSTRATIONS

1. $\sqrt{9 \times 4} = \sqrt{36} = 6$ and $\sqrt{9} \times \sqrt{4} = 3 \times 2 = 6$.
2. $\sqrt{\frac{144}{36}} = \sqrt{4} = 2$ and $\frac{\sqrt{144}}{\sqrt{36}} = \frac{12}{6} = 2$.

(b) Rationalisation of Radical (or Surd) Expressions.

It is often desirable to eliminate expressions containing radicals from the denominator of a quotient. This process is called

rationalising the denominator.

The process involves multiplying numerator and denominator of the quotient by the same amount - an amount which eliminates the radicals in the denominator (often using the fact that the square root of a number multiplied by itself gives just the number;
i.e. $\sqrt{a}.\sqrt{a} = a$). We illustrate with examples:

EXAMPLES

1. Rationalise the surd form $\frac{5}{4\sqrt{3}}$

   Solution

   We simply multiply numerator and denominator by $\sqrt{3}$ to give

$$\frac{5}{4\sqrt{3}} = \frac{5}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{12}.$$

2. Rationalise the surd form $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$

   Solution

   Here we observe that, if we can convert the denominator into the cube root of b^n, where n is a whole multiple of 3, then the square root sign will disappear.

   We have

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}} \times \frac{\sqrt[3]{b^2}}{\sqrt[3]{b^2}} = \frac{\sqrt[3]{ab^2}}{\sqrt[3]{b^3}} = \frac{\sqrt[3]{ab^2}}{b}.$$

   If the denominator is of the form $\sqrt{a} + \sqrt{b}$, we multiply the numerator and the denominator by the expression $\sqrt{a} - \sqrt{b}$ because

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

3. Rationalise the surd form $\frac{4}{\sqrt{5}+\sqrt{2}}$.

   Solution

   Multiplying numerator and denominator by $\sqrt{5} - \sqrt{2}$ gives

$$\frac{4}{\sqrt{5} + \sqrt{2}} \times \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}} = \frac{4\sqrt{5} - 4\sqrt{2}}{3}.$$

4. Rationalise the surd form $\frac{1}{\sqrt{3}-1}$.

   Solution

   Multiplying numerator and denominator by $\sqrt{3}+1$ gives

$$\frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2}.$$

(c) Changing numbers to and from radical form

The modulus of any number of the form $a^{\frac{m}{n}}$ can be regarded as the principal n-th root of a^m; i.e.

$$\mid a^{\frac{m}{n}} \mid = \sqrt[n]{a^m}.$$

If a number of the type shown on the left is converted to the type on the right, we are said to have expressed it in radical form.

If a number of the type on the right is converted to the type on the left, we are said to have expressed it in exponential form.

Note:
The word **"exponent"** is just another word for **"power"** or **"index"** and the standard rules of indices will need to be used in questions of the type discussed here.

EXAMPLES

1. Express the number $x^{\frac{2}{5}}$ in radical form.

   Solution

   The answer is just

$$\sqrt[5]{x^2}.$$

2. Express the number $\sqrt[3]{a^5b^4}$ in exponential form.

   Solution

   Here we have

$$\sqrt[3]{a^5b^4} = (a^5b^4)^{\frac{1}{3}} = a^{\frac{5}{3}}b^{\frac{4}{3}}.$$

1.3.3 EXERCISES

1. Simplify
   (a) $5^7 \times 5^{13}$; (b) $9^8 \times 9^5$; (c) $11^2 \times 11^3 \times 11^4$.
2. Simplify
   (a) $\frac{15^3}{15^2}$; (b) $\frac{4^{18}}{4^9}$; (c) $\frac{5^{20}}{5^{19}}$.
3. Simplify
   (a) a^7a^3; (b) a^4a^5;
   (c) $b^{11}b^{10}b$; (d) $3x^6 \times 5x^9$.
4. Simplify
   (a) $(7^3)^2$; (b) $(4^2)^8$; (c) $(7^9)^2$.
5. Simplify
   (a) $(x^2y^3)(x^3y^2)$; (b) $(2x^2)(3x^4)$;
   (c) $(a^2bc^2)(b^2ca)$; (d) $\frac{6c^2d^3}{3cd^2}$.
6. Simplify
   (a) $\left(4^{-3}\right)^2$ (b) $a^{13}a^{-2}$;
   (c) $x^{-9}x^{-7}$; (d) $x^{-21}x^2x$;
   (e) $\frac{x^2y^{-1}}{z^3} \div \frac{z^2}{x^{-1}y^3}$.
7. Without using a calculator, evaluate the following:
   (a) $\frac{4^{-8}}{4^{-6}}$; (b) $\frac{3^{-5}}{3^{-8}}$.
8. Evaluate the following:
   (a) $64^{\frac{1}{3}}$; (b) $144^{\frac{1}{2}}$;
   (c) $16^{-\frac{1}{4}}$; (d) $25^{-\frac{1}{2}}$;
   (e) $16^{\frac{3}{2}}$; (f) $125^{-\frac{2}{3}}$.
9. Simplify the following radicals:
   (a) $-\sqrt[3]{-8}$; (b) $\sqrt{36x^4}$; (c) $\sqrt{\frac{9a^2}{36b^2}}$.
10. Rationalise the following surd forms:
    (a) $\frac{\sqrt{2}}{\sqrt{3}}$; (b) $\frac{\sqrt[3]{18}}{\sqrt[3]{2}}$; (c) $\frac{2+\sqrt{5}}{\sqrt{3}-2}$; (d) $\frac{\sqrt{a}}{\sqrt{a}+3\sqrt{b}}$.
11. Change the following to exponential form:
    (a) $\sqrt[4]{7^2}$; (b) $\sqrt[5]{a^2b}$; (c) $\sqrt[3]{9^5}$.

12. Change the following to radical form:
    (a) $b^{\frac{3}{5}}$; (b) $r^{\frac{5}{3}}$; (c) $s^{\frac{7}{3}}$.

1.3.4 ANSWERS TO EXERCISES

1. (a) 5^{20}; (b) 9^{13}; (c) 11^9.
2. (a) 15; (b) 4^9; (c) 5.
3. (a) a^{10}; (b) a^9; (c) b^{22}; (d) $15x^{15}$.
4. (a) 7^6; (b) 4^{16}; (c) 7^{18}.
5. (a) x^5y^5; (b) $6x^6$; (c) $a^3b^3c^3$; (d) $2cd$.
6. (a) 4^{-6}; (b) a^{11}; (c) x^{-16}; (d) x^{-18}; (e) xy^2z^{-5}.
7. (a) $\frac{1}{16}$; (b) 27.
8. (a) 4; (b) ± 12; (c) $\pm\frac{1}{2}$;
   (d) $\pm\frac{1}{5}$; (e) ± 64; (f) $\frac{1}{25}$;
9. (a) 2; (b) $6x^2$; (c) $\left|\frac{a}{2b}\right|$.
10. (a) $\frac{\sqrt{6}}{3}$; (b) $\frac{\sqrt[3]{72}}{2} = \sqrt[3]{9}$; (c) $-(2+\sqrt{5})(2+\sqrt{3})$; (d) $\frac{a-3\sqrt{ab}}{a-9b}$.
11. (a) $\left|7^{\frac{1}{2}}\right|$; (b) $a^{\frac{2}{5}}b^{\frac{1}{5}}$; (c) $9^{\frac{5}{3}}$.
12. (a) $\sqrt[5]{b^3}$; (b) $\sqrt[3]{r^5}$; (c) $\sqrt[3]{s^7}$.