Simple connectedness of space-time in the path topology

Robert J Low*
Department of Mathematics, Statistics,
and Engineering Science,
Coventry University,
Coventry CV1 5FB,
UK

Abstract

We extend earlier work on the simple-connectedness of Minkowksi space in the path topology of Hawking, King and McCarthy, showing that in general a space-time is neither simply connected nor locally simply connected. We also show that distinct closed Feynman paths are never homotopically equivalent, so that space-time is in a sense as non-simply connected as possible.

PACS numbers: 02.40.Pc, 04.20.Gz

keywords: general relativity, topology, causal structure.

*email: mtx014@coventry.ac.uk

1 Introduction and Context

The use of non-standard topologies to encode various structures on space-time has resulted in some interesting approaches to the study of space-time geometry.

Zeeman [1] showed that the fine topology on Minkowski space (the finest topology which induces the standard topology on each timelike and spacelike hypersurface) encodes the information of the causal and linear structure of Minowski space. However, this topology is technically very complicated and difficult to calculate with—it does not admit a neighboughood basis. It is also strongly tied to the linear structure of Minkowski space, which makes it of restricted interest in general relativity.

The fine topology was subsequently generalised to the case of a curved space-time by Göbel [4], by taking it to be the finest topology with respect to which all timelike geodesics and spacelike hypersurfaces are continuous. The generalised topology encodes much interesting information about the original space-time, as it specifies the metric up to an overall constant: however, it remains technically difficult to work with. It also has some properties which one might regard as undesirable. An obvious example is that although timelike geodesics are continuous, general timelike curves are not.

An alternative topology, the path topology, was considered by Hawking, King and McCarthy [2]. The path topology is defined to be the finest topology in which all timelike curves are continuous. This topology turns out to be much more tractable than the fine topology. For if M is a space-time, then the path topology of M (as is not the situation for even Minkowski space in the fine topology) can be described in terms of a neighbourhood basis. For each $x \in M$, and each open neighbourhood U of x, we denote by $I(p, U)$ the set of points connected to p by a timelike curve lying in U, and by $K(p, U)$ the set $I(p, U) \cup \{x\}$. Choose some arbitrary Riemannian metric h on M, and denote by $B_\epsilon(x)$ the open ball of all points within distance ϵ of x with respect to h. Finally, for an open convex normal neighbourhood, U, of x, let $L_U(x, \epsilon) = K(p, U) \cap B_\epsilon(x)$. The sets of the form $L_U(\epsilon, x)$ are then a basis for the path topology. Indeed, by choosing $\epsilon = 1/n$ for $n \in \mathbb{N}$, we obtain a countable neighbourhood basis, so that the path topology is first countable.

The path topology on M is of great physical interest. The continuous curves are precisely the Feynman paths, and the path topology induces the discrete topology on null and spacelike sets. Furthermore, the path topology determines both the differentiable and conformal structure of M. On the other hand, as it only specifies the metric up to a conformal factor, it is less discriminatory than the fine topology. However, in compensation for this, the path topology on M is much simpler and easier to work with than the fine topology, and it is interesting to investigate other properties of this topology on space-time.

Many properties of the path topology were already established in [2]: it is strictly finer than the manifold topology, first countable, separable, Hausdorff, path connected, and locally path connected. (Definitions of these various conditions can be found in, for example [3].)

Recently, Dossena in [5] revisited the fine topology on Minkowski space and considered connectedness properties, showing in particular that the fundamental group of two dimensional Minkowski space with the fine topology has uncountably many subgroups isomorphic to $\mathbb{Z}$. The arguments made use of Zeno sequences, a Zeno sequence being a sequence which converges in the manifold topology but not the fine topology. Agrawal and Shrivastava applied similar arguments to consider the path topology of two dimensional Minkowski space in [6], and showed that in this topology also, two dimensional Minkowski space fails to be simply connected.

In the following section we will see how to extend the results of [6] to any dimension, and to general curved space-times, while making use of simpler arguments which do not require the use of Zeno sequences.

2 Simple-connectedness of space-time in the path topology

So let $\mathbb{M}$ denote Minkowski space with the usual (t, x, y, z) coordinates with respect to which the metric is $g = \text{diag}(1, -1, -1, -1)$. Now fix p, q and r so that a future pointing timelike vector connects p to q, another connects q to r, and a third connects p to r. Assume that p, q and r all lie in the (t, x) plane. Denote by pq, qr and rq the lines in $\mathbb{M}$ connecting the points, and by $pqrp$ the union of the three lines. Without loss of generality, we can assume that p has coordinates $(-1, 0, 0, 0)$ and r has coordinates $(1, 0, 0, 0)$.

Denote $[0, 1]$ by I. Let $h : I \times I \to \mathbb{M}$ and denote by h_s the mapping given by $h_s(t) = h(s, t)$. Suppose that h also has the properties that

1. h_0 has range $pqrp$
2. h_1 has range p,

so that h is a homotopy between $pqrp$ and p.

Theorem 1 The function h, as defined above, cannot be continuous in the path topology.

Proof Suppose that h is in fact continuous in the path topology. Then, since the path topology is finer than the manifold topology, h must be continuous in the manifold topology on $\mathbb{M}$. We now consider the situation in the manifold topology of $\mathbb{M}$.

Denote by J the image of h in $\mathbb{M}$. Since h is continuous and $I \times I$ is compact, J must be compact, and since $\mathbb{M}$ is Hausdorff, J is therefore closed. (See [7] for those results in elementary topology used here.)

Let π be the projection from $\mathbb{M}$ to the timelike plane $T = \{(t, x, 0, 0)\}$. Since h is continuous, so is $\pi \circ h$, which then maps $I \times I$ to T. Now, let x be a point inside $pqrp$ in T. Then, as in [6], x must be in the image of $\pi \circ h$, since $h(0, .)$ maps I to $pqrp$ which has winding number 1 round x, and $h(1, .)$ maps I to p, which has winding number 0 round x. (See [8] for an extensive discussion of winding number and its properties.)

Now, there are infinitely many points inside T with t coordinate 0, and so the intersection of J with the surface $\Sigma_0 = \{(0, x, y, z)\}$ contains infinitely many points. But since h is continuous, and $I \times I$ is compact, J is compact. Since Σ_0 is closed, $J_0 = \Sigma_0 \cap J$ is compact and closed. But since J_0 is closed, $h^{-1}(J_o)$ is closed, and so, since $I \times I$ is compact, so is $h^{-1}(J_0)$.

We now consider J_0 in the path topology. Since J_0 is acausal, it inherits the discrete topology, and since it contains infinitely many points, it cannot be compact. But if h is continuous, we have $J_0 = h(h^{-1}(J_0)$, the continuous image of a compact set, and so J_0 must be compact.

This contradiction shows that h cannot be continuous. □

It is worth considering one particular obvious candidate h for a homotopy. Choose some continuous (in the path topology) parameterisation of $pqrp$, and take h_0 to be this paramaterisation. Then let $f(t, s) = tp + (1 - t)h_0(s)$, so that as t increases from 0 to 1 each point of $pqrp$ is carried to p along a timelike line. Furthermore, for each t, the image of h_t is a scaled copy of $pqrp$, and hence each h_t is continuous in the path topology. This therefore provides a (rather exotic) example of a function of two variables which is continuous in each variable separately, but is not in fact continuous at any point of its domain.

We also note that in the above argument, it was irrelevant that the dimension of Minowski space was 4. The proof is identical for Minkowski space of any dimension greater than or equal to 2. Denote by $\mathbb{M}^n$ Minkwoski space of n dimensions.

We thus obtain

Corollary 1 $\mathbb{M}^n$ is not simply connected in the path topology for any $n \geq 2$. □

Furthermore, observing that we can have q and r lying in arbitrarily small (path topology) neighbourhoods of p, we have

Corollary 2 $\mathbb{M}^n$ is not locally simply connected for any $n \geq 2$. □

This shows that in the path topology, no continuous closed curve in $\mathbb{M}$ is null-homotopic. But we can adapt the argument to show that no two distinct closed continuous curves in $\mathbb{M}$ are homotopic.

Theorem 2 If c_1 and c_2 are closed continuous maps from S^1 to $\mathbb{M}$ with distinct images, then c_1 and c_2 are not homotopic in the path topology.
Proof Let c_1 and c_2 be curves in $\mathbb{M}$ with distinct images, let $h : I \times I \to \mathbb{M}$ such that $h(0, .)$ is c_1 and $h(1, .)$ is c_2, and let T be some timelike two-plane and associated projection π such that the projections of c_1 and c_2 to T are distinct. First, we note that neither of $\pi \circ c_1$ nor $\pi \circ c_2$ can be space-filling, for then we already have an open set in T containing infinitely many points in some spacelike surface and in the image of $\pi \circ h$, which allows us to argue as before by considering the intersection of this open set with some surface of constant time.

So there must now be some point x in T round which c_1 and c_2 have different winding numbers. Since c_1 and c_2 are closed curves in T, x has an open neighbourhood in T which lies in the image of $\pi \circ h$, and again we obtain a contradiction as in the proof to Theorem 1. □

We see, then that the fundamental group of $\mathbb{M}$ with the path topology is as large as possible, since two continuous loops are only homotopic is one is a reparameterisation of the other.

Finally, we can see that the above results pass to the case of a general Lorentz manifold, courtesy of the isometric embedding theorems [9]. For the general space-time M, we embed it in a pseudo-Euclidean space of appropriate dimension, and proceed just as before by projecting to some suitable timelike plane in the pseudo-Euclidean space. We thus obtain:

Theorem 3 A space-time, M, equipped with the path topology is not simply connected or locally simply connected. Furthermore, no two closed continuous curves in M with distinct images are homotopic. □

As is the case for Minkowski space with the fine topology, $\pi_1(M)$ has uncountably many subgroups isomorphic to $\mathbb{Z}$, since each distinct continuous loop based at some $x \in M$ gives a distinct subgroup. In fact, we can determine the cardinality of the set of subgroups easily.

First, there is at least one continuous loop for each real number, so that cardinality of the set of subgroups is at least that of the continuum. Second, since a continuous function with domain S^1 is determined by its values on points of S^1 with a rational coordinate, the cardinality of this set of functions is at most that of the continuum. Hence $\pi_1(M)$ has as many subgroups isomorphic to $\mathbb{Z}$ as there are real numbers.

We see, then, that the fundamental group of any space-time in the path topology is extremely large, since distinct continuous loops are never homotopic. This is essentially a consequence of the fact that the path topology induces the discrete topology on null and spacelike subsets of M, since any non-trivial homotopy (in the standard topology) has in its image an infinite spacelike set which is the image of a compact subset of $I \times I$, thus preventing the homotopy from being continuous when the path topology on M is used.

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