Comment on 'A note on non-compact Cauchy surfaces'

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Abstract

Kim [1] has recently shown how to reconstruct a globally hyperbolic space-time with non-compact Cauchy surface Σ up to a conformal factor by considering a family of subsets of Σ. We see how this work is related to previous results on reconstructing such space-times up to a conformal factor from the set of skies in the space of null geodesics.

Let M be a globally hyperbolic space-time with non-compact Cauchy surface Σ. Then there are two families of compact subsets of Σ given by sets of the form $J^-(p) \cap \Sigma$ and $J^+(p) \cap \Sigma$ for $p \in M$. These sets are the causally admissible subsets of Σ, and $J(p) \cap \Sigma$ is denoted S_p. Kim has shown [1] that the causally admissible sets encode the conformal structure of M.

First, we note that any causally admissible subset of Σ, S_p, is uniquely determined by its boundary, ∂S_p.

Next, recall that $\mathcal{N}$, the space of null geodesics of M, may be identified with $T^*S(\Sigma)$, the cotangent sphere bundle of Σ, and that we can associate with any $p \in M$ its sky $P \subset \mathcal{N}$ which is the set of all null geodesics passing through p in M; we can also associate with p the set $N(p)$ in M consisting of all points connected to p by a null geodesic. It is already known [2] that knowledge of the skies in $\mathcal{N}$ allows the original space-time to be reconstructed up to a conformal factor in the case where M is globally hyperbolic with non-compact Cauchy surface.

For a point p not on Σ, we can then consider the set $N(p) \cap \Sigma$, which is the image in Σ under a smooth map of a Legendrian S^2 in $\mathcal{N}$, and we can attach to this surface

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the projection to $T\Sigma$ of the future pointing null congruence tangent to $N(p)$ at Σ. This gives a representation of P as a subset of Σ, equipped with an orthogonal vector field.

Now, note that $\partial S_p \subseteq N(p) \cap \Sigma$. We can therefore lift ∂S_p to $\mathcal{N}$ by associating with it an orthogonal congruence tangent to Σ, inward pointing if p is from the a point to the future of Σ and past pointing if it is from a point to the past, then lifting each point of ∂S_p to the null geodesic whose future pointing tangent has the appropriate projection to $T\Sigma$. Note that the distinction between the two compact subsets of Σ is precisely the distinction between the inward and outward pointing orthogonal congruence.

Thus each causally admissible set S_p, together with the information of which family it belongs to, specifies a subset of P; for a general $p \in M$, denote this subset of P by P' (noting that it depends on the choice of Cauchy surface). P' will equal P itself if p is sufficiently close to Σ, but in general only includes those points of Σ connected to p by a maximal null geodesic. Geodesics are not included if they only reach Σ after a passing through a conjugate point or intersecting another null geodesic through p.

We therefore see that in this framework Kim's result can be understood as saying that full knowledge of the skies of all points is unnecessary for the reconstruction of space-time; it is enough to have, for each point p, the portions corresponding to those null geodesics which have not yet entered $I(p)$ *en route* to Σ.

So for a globally hyperbolic space-time with a non-compact Cauchy surface, this strengthens the previous result on reconstructing space-time from the set of skies, by showing that a strict subset of the information suffices. Indeed, one can see intuitively how this works in the framework of skies in $\mathcal{N}$ as follows. If U is a small neighbourhood of p, and P' is the subset of P corresponding to $\partial J(p) \cap \Sigma$, then for each point q in U, the corresponding subset Q' of Q will be close to P', and will intersect Q' if and only if p and q lie on each other's light cone: this shows that the null geodesics and hence the metric up to a conformal factor are encoded.

One might therefore ask if this approach has any disadvantages to accompany this efficiency gain. In fact, we can see that it does.

Kim observes [1] that there are examples of space-times with compact Cauchy surface for which the causally admissible subsets of Σ cannot allow one to reconstruct M; the Einstein cylinder is the obvious example, as given any Cauchy surface Σ there are many points p for which $S_p = \Sigma$.

On the other hand, the use of the complete sky allows one to reconstruct the space-time in many cases where the Cauchy surface Σ is compact. Chernov and Rudyak have shown [3] that the set of skies in $\mathcal{N}$ encodes M up to a conformal factor as long as the uiversal covering space of Σ is non-compact.

However, the approach using causally admissible sets certainly fails in at least some of these cases: for example, let $M = \mathbb{R} \times \mathbb{T}^3$, a static space-time with Σ the standard flat torus. Again, given any choice of Σ there are many points p for which $S_q = \Sigma$, and so the space-time cannot be reconstructed.

In spite of this, we can extend the result to at least some space-times with a compact Cauchy surface.

Theorem Let M be a globally hyperbolic space-time with Cauchy surface Σ. Then provided that Σ is not itself a causally admissible subset of Σ, the causally admissible subsets specify M up to a conformal factor.

Proof First, we note that if Σ is a causally admissible subset, the approach fails. For if $S_p = \Sigma$, then whenever $q \in J^+(p)$, $S_q = \Sigma$, and one does not even recover M as a point set.

Now suppose that for all p, S_p is a strict subset of Σ. Then the arguments already presented by Kim proceed as before, and M can be reconstructed from the set of causally admissible subsets of Σ. $\square$

References

[1] Kim D-H (2008) A note on non-compact Cauchy surfaces *Classical and Quantum Gravity* **25** 238002

[2] Low RJ (2001) The space of null geodesics *Nonlinear Analysis* **47** 30053017

[3] Chernov VV & Rudyak YB (2008) Linking and causality in globally hyperbolic space-times *Communications in Mathematical Physics* **279** 309–354