Characterizing the Causal Automorphisms of 2-d Minkowski space

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Abstract

We present a simple characterization of the causal automorphisms of 2-d Minkowski space, and relate it to the characterization provided by Kim [1]

1 Introduction

Let M be a space-time, i.e. a smooth Lorentz manifold. For $x, y \in M$ we say that x chronologically precedes y, denoted $y \in I^+(x)$ if there is a smooth, future directed, timelike curve from x to y, and that x causally precedes y, $y \in J^+(x)$, if there is a smooth, future directed, causal curve from x to y [2]. A bijection from M to itself which preserves these causal relations is called a causal automorphism. The set of causal automorphisms is clearly a group under composition, and can therefore be thought of as the symmetry group of the causal structure of M.

It has been known since 1964 that in more than two space-time dimensions the causal automorphisms of Minkowski space are generated by the inhomogeneous Lorentz group together with the dilatations [3]. However, this is not the case in two space-time dimensions. More recently, in 2009, Kim [4] showed that the group of causal automorphisms of two-dimensional Minkowski space, $\mathbb{M}^2$, was infinite-dimensional, and then in 2010 Kim [1] also showed that it was characterized by two functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfying the rather non-obvious conditions that

1. f is a homeomorphism

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2. g is continuous

3. $\sup(g \pm f) = \infty$

4. $\inf(g \pm f) = -\infty$

5. $\left|\frac{g(t+\delta t)-g(t)}{f(t+\delta t)-f(t)}\right| < 1$ for all t and δt.

A causal automorphism is specified by such a pair f, g as follows: put the usual Minkowskian coordinates (x, t) on $\mathbb{M}^2$, and send the point $(x, 0)$ to $(f(x), g(x))$. The constraints on f and g ensure that this is again a Cauchy surface. By considering causally admissible sets—a subset of a Cauchy surface is causally admissible if it comprises all the points causally related to some point in space-time—Kim shows that this then determines a unique causal automorphism on $\mathbb{M}^2$, and that all causal automorphisms are obtained in this way. Specifically, if (x, t) are the usual Minkowskian coordinates on $\mathbb{M}^2$, then the point (x, t) is mapped to (X, T) where

$$\begin{aligned} X &= \frac{f(x+t)+g(x+t)}{2} - \frac{g(x-t)-f(x-t)}{2} \\ T &= \frac{f(x+t)+g(x+t)}{2} + \frac{g(x-t)-f(x-t)}{2} \end{aligned} \tag{1}$$

The purpose of this article is to show how the standard null coordinates on two-dimensional Minkowski space provide a more perspicuous characterization of the causal automorphisms on Minkowski space, and to investigate the relationship between the two characterizations.

2 Characterization

First, let (x, t) be the usual Minkowksian coordinates on two-dimensional Minkowski space, $\mathbb{M}^2$, and define null coordinates (u, v) by $u = t + x$, $v = t - x$.

We will show that $\phi : \mathbb{M}^2 \to \mathbb{M}^2$ is a causal automorphims if and only if it is of one of the forms

$$(u, v) \mapsto (U(u), V(v)) \text{ or } (u, v) \mapsto (U(v), V(u))$$

where U and V are both order-preserving bijections.

First, we observe that an order-preserving bijection $\mathbb{R} \to \mathbb{R}$ is necessarily a homeomorphism. For it is easy to see that an order-preserving bijection is continuous, and then (by invarance of domain [5]) it is a homeomorphism.

So now we consider the case of a mapping of the first form above,

$$(u, v) \mapsto (U(u), V(v)),$$

which is the case of an orientation preserving causal automorphism of $\mathbb{M}^2$. If we denote $U(u)$ and $V(v)$ by U and V respectively, then (abusing notation by identifying points in

$\mathbb{M}^2$ with their coordinates) we immediately see that

$$\begin{aligned}(U_1, V_1) \in I^+(U_2, V_2) &\Leftrightarrow U_1 > U_2 \wedge V_1 > V_2 \\ &\Leftrightarrow u_1 > u_2 \wedge v_1 > v_2 \\ &\Leftrightarrow (u_1, v_1) \in I^+(u_2, v_2)\end{aligned}$$

and hence the mapping is a causal automorphism.

An analogous argument establishes that mappings of the second type, which reverse the orientation of $\mathbb{M}^2$, are also causal automorphisms.

Next, suppose that $\phi : \mathbb{M}^2 \to \mathbb{M}^2$ is a causal automorphism. Then ϕ is a bijection which maps Alexandrov open sets to Alexandrov open sets. Since $\mathbb{M}^2$ is strongly causal, the Alexandrov topology coincides with the manifold topology [6]. Hence ϕ is continuous and open, and so is a homeomorphism.

Now, since the null geodesics of $\mathbb{M}^2$ bound the future and past sets, the mapping ϕ must preserve them. If follows that ϕ must be of one of the forms

$$(u, v) \mapsto (U(u), V(v)) \text{ or } (u, v) \mapsto (U(v), V(u)).$$

Since ϕ is a bijection, each of U and V must be a bijection, and since ϕ preserves the causal ordering, each of U and V must be order preserving, and so must be continuous.

Thus we have a simple characterization of a causal automorphism of $\mathbb{M}^2$: it is specified by a pair of order-preserving bijections on $\mathbb{R}$.

3 Correspondence

So we must ask, how do this U and V relate to the functions f and g specified by Kim?

We must re-express in terms of (x, t) the causal automorphism in terms of (u, v). Again, we consider explicitly the case where orientation is preserved, which is the case where f is increasing. This yields

$$\begin{aligned}X &= \frac{U - V}{2} = \frac{1}{2}\left[U(t + x) - V(t - x)\right] \\ T &= \frac{U + V}{2} = \frac{1}{2}\left[U(t + x) + V(t - x)\right]\end{aligned} \tag{2}$$

Now recall equation (1), which tells us that the functions f and g give the causal automorphism via

$$\begin{aligned}X &= \frac{f(x + t) + g(x + t)}{2} - \frac{g(x - t) - f(x - t)}{2} \\ T &= \frac{f(x + t) + g(x + t)}{2} + \frac{g(x - t) - f(x - t)}{2}\end{aligned}$$

Comparing (1) and (2) we see that

$$\begin{aligned}g(z) + f(z) &= U(z) \\ g(z) - f(z) &= V(-z).\end{aligned}$$

so that

$$g(z) = U(z) + V(-z)$$
$$f(z) = U(z) - V(-z).$$

This yields

$$\frac{g(z+\delta z) - g(z)}{f(z+\delta z) - f(z)} = \frac{[U(z+\delta z) - U(z)] - [V(-z) - V(-z-\delta z)]}{[U(z+\delta z) - U(z)] + [V(-z) - V(-z-\delta z)]}.$$

We therefore see that the conditions Kim finds on f and g are equivalent to the conditions that U and V be increasing, continuous, and onto.

The case where f is decreasing is entirely analogous.

4 Conclusions

1. There is a simple characterization of causal automorphisms in terms of null coordinates.
2. Kim's conditions have a natural interpretation when the mapping they represent on $\mathbb{M}^2$ is represented in null coordinates, and in that framework it is easy to see why they correspond to a causal automorphism.

Comment: It is also worth observing that by considering the situation in terms of Cartesian coordinates, we can see that X and T are both given by solutions of the wave equation on $\mathbb{M}^2$ (at least in the case where they are sufficiently differentiable). It would be interesting to know whether there is a useful characterization of just which solutions of the wave equation give rise to causal automorphisms of $\mathbb{M}^2$.

Acknowledgement

I thank the anonymous referees for their help in improving the presentation of this paper.

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