Causality and Borromean Linking in Twistor Space

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Abstract

We consider the notion of the Borromean triple from classical linking theory, and develop an analogue of it in the case of linking of skies in twistor space. I would like to dedicate this paper to Roger Penrose, on the occasion of his 80th birthday.

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1 Introduction and Context

Recall that in the case of Minkowski space-time, there is a simple and elegant relationship between the casual separation of points, $x, y \in \mathbb{M}$, and the linking of the corresponding skies $X, Y \subset \mathbb{PN}^I$, where $\mathbb{PN}^I$ is the space of null geodesics of $\mathbb{M}$, which has topology $\mathbb{R}^3 \times S^2$ [1]. By an appropriate choice of sign conventions and orientations, one can define a linking number, $\mathrm{Lk}(X, Y)$ such that one has

$$y \in I^+(x) \text{ if and only if } \mathrm{Lk}(X, Y) = 1.$$

Furthermore, the linking number can be computed by taking a Cauchy surface, $\mathcal{S}$, in $\mathbb{M}$ which contains x and calculating the winding number of the intersection of y's light cone with $\mathcal{S}$ around x.

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Any null geodesic γ in $\mathbb{M}$ intersects $\mathcal{S}$ in exactly one point, $\gamma_{\mathcal{S}}$, and we can take the direction in $TS(\mathcal{S})$, the tangent sphere bundle to $\mathcal{S}$ with its induced Riemannian metric, to which the future pointing tangent to γ at $\gamma_{\mathcal{S}}$ projects. This allows us to identify $\mathbb{PN}^I$ with $TS(\mathcal{S})$, and we can then see that x and y are chronologically related in $\mathbb{N}$ precisely when it is impossible to deform Y to a fibre of $TS(\mathcal{S})$ by an isotopy in $TS(\mathcal{S} \setminus \{x\})$.

This result does not extend to more general space-times. However, one can introduce the more delicate notion of Legendrian isotopy. This requires that the surface at each stage be a Legendrian submanifold of the space of null geodesics endowed with it natural contact structure. In terms of the original space-time, this entails that the null geodesics generate a deformed null cone in such a way that the intersection of this deformed null cone with any smooth spacelike surface is orthogonal to the null generators. Nemurowski and Chernov [2] have shown that if one uses this notion of isotopy, then in a very wide class of globally hyperbolic space-times, the result holds.

It is tempting then to look for analogues to interesting examples of the classical case of linking, namely that of S^1's in $\mathbb{R}^3$ (or S^3).

One such interesting example is the Borromean triple [3]: a set of three loops in $\mathbb{R}^3$, such that any two are unlinked in $\mathbb{R}^3$, but are linked in the complement of the third (Figure 1). One might then ask whether there is

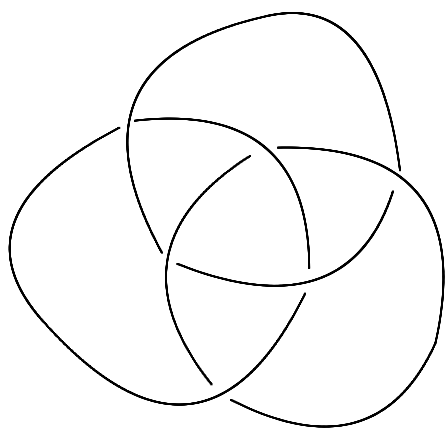

Figure 1: Borromean Rings

an analogue to this in the case of linking of skies in twistor space, and, if so, what kind of causal relationship it describes.

First, we have to describe the Borromean configuration in a suitable way.

Note that two spheres X and Y in Euclidean space are unlinked if they can be deformed by a simultaneous isotopy to a pair of spheres on opposite sides of a hyperplane, or, equivalently, if X can be deformed by an isotopy (not intersecting Y) to a sphere which is separated from Y by a hyperplane. In the case of linking of skies in $\mathbb{PN}^I$, this corresponds to the isotopic deformation of X and Y to two distinct fibres of $\mathbb{PN}^I$ considered as $TS(\mathcal{S})$ where $\mathcal{S}$ is some Cauchy surface for $\mathbb{M}$.

Then three skies will form a Borromean triple if any two of them can be simultaneously be isotopically deformed to fibres, but this deformation necessarily intersects the third sky at least once.

2 No Borromean triples for Minkowski space-time

First, we can argue simply that there can be no Borromean triple of skies for the case of Minkowski space-time.

For let X, Y, Z be three skies such that the link of any pair is trivial. Then any pair of the three points x, y, z in $\mathbb{M}$ is spacelike separated. It does not necesarily follow that the three points must lie in a common surface of constant t. However, any two of them must, so let us suppose that (with the usual coordinates on Minkowski space) the point x and y both lie in the Cauchy surface $\mathcal{S}$, which is given by $t = 0$.

Now, if z also lies in $\mathcal{S}$, we are done, so suppose not. Suppose that z is to the future of $\mathcal{S}$, and let γ be a future pointing curve from some point of $\mathcal{S}$ to z. See Figure 2 for a sketch of this situation. Then γ cannot intersect the light cone of x, for if it does, z is to the chronological future of a point to the causal future of x, and so x is in the chronological future of x. Similarly, this curve cannot cross the light cone of z. But the curve gives an isotopic deformation of Z in $\mathbb{PN}^I$ which does not meet either X or Y, and carries Z so that X, Y and Z are all fibres of $\mathbb{PN}^I$. If z is to the past of $\mathcal{S}$, the situation is similar.

Hence, there is no analogue to the Borromean triple for linking of skies in the case of Minkowski space-time.

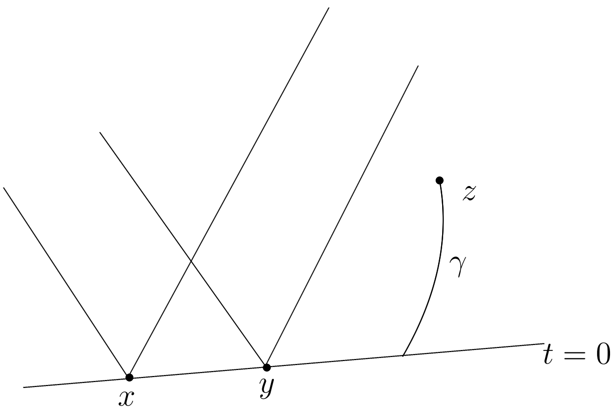

Figure 2: Minkowski space-time

3 No Borromean triples for globally hyperbolic space-times

Now, let M be a globally hyperbolic space-time satisfying the conditions of Nemirowski and Chernov [2], with Cauchy surfaces $\mathcal{S}_t$, and time function t (so that the curves with tangent vector $\partial/\partial t$ give a congruence of timelike curves, $\mathcal{C}$, filling M). Next let x, y, z be points such that X, Y and Z are pairwise unlinked, supposing without loss of generality that x lies in the Cauchy surface $\mathcal{S}_{t_0}$, y lies in $\mathcal{S}_{t_1}$ and z lies in $\mathcal{S}_{t_2}$, where $t_0 \leq t_1 \leq t_2$. We consider the space of null geodesics, $\mathcal{N}$, as given by the tangent sphere bundle over $\mathcal{S}_{t_0}$.

First, drag z to the past along the timelike curve γ_z in $\mathcal{C}$ through z, until it reaches $z' \in \mathcal{S}_{t_1}$. At no point in this motion can the curve meet the light cones of either y or x, or by the argument above, z must lie to the future of either x or y. Next, take y and z simultaneously along the timelike curves γ_y and γ_z respectively through y and z' in $\mathcal{C}$ until they reach $\mathcal{S}_{t_0}$. Since y and z are in the same Cauchy surface throughout this, neither passes through the light cone of the other: and as before, neither curve can pass through the light cone of x. A sketch of this situation is shown in Figure 3.

Hence in $\mathcal{N}$ we obtain an isotopic deformation that takes Y and Z to fibres, without passing through X. Furthermore, since the isotopy arises from a curve in M, it is in fact an isotopy through skies, and so is a Legendrian isotopy. Thus there is no analogue to a Borromean triple in a globally hyperbolic space (where Legendrian linking corresponds to causal separation).

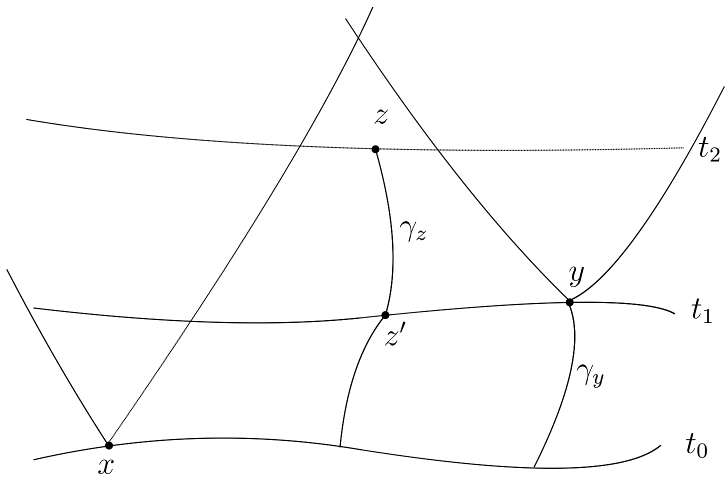

Figure 3: Globally hyperbolic space-time

4 Borromean Triples in compactified Minkowski space

Surprisingly, if we start instead in a causally ill-behaved space-time—in fact a totally vicious space-time—then we can obtain a Borromean triple which has an interpretation in terms of the causal and topological structure of space-time.

So let $\mathbb{M}^{\#}$ be the standard compactified and identified Minkowksi space [4], diffeomorphic with $S^3 \times S^1$. This is Minkowski space together with a point i at infinity and its null cone, all identified to provide a compact manifold without boundary. $\mathbb{M}^{\#}$ can be given a convenient visualisation in terms of the conformal embedding of $\mathbb{M}$ into the Einstein cylinder, which is given by $S^3 \times \mathbb{R}$, where S^3 is a standard unit sphere [5]. First, identify t with $t + 2\pi$ to give the cyclic Einstein space. Next, identify (p, t) with $(-p, t + \pi)$, to obtain a space again diffeomorphic to $S^3 \times S^1$. The Einstein cylinder is thus the universal covering space of $\mathbb{M}^{\#}$. Also denote the closure of (a copy of) Minkowski space conformally embedded into the Einstein cylinder with its distinct points at future infinity i^+, spacelike infinity i^0, and past infinity i^- by $\overline{\mathbb{M}}$: these points are identified to i in $\mathbb{M}^{\#}$.

Note that $\mathbb{M}^{\#}$ is totally vicious—any two points can be connected by a smooth curve whose tangent is everywhere timelike and future pointing. Note also that removing any point and its light cone from $\mathbb{M}^{\#}$ recovers a space conformal to $\mathbb{M}$: by an appropriate conformal transformation, we can choose the removed point to be the point at infinity.

Furthermore, the space of all null geodesics of $\mathbb{M}^\#$ is projective null twistor space, $\mathbb{PN}$, with topology $S^3 \times S^2$ [4]. This can be again identified with $TS(\mathcal{S})$ where $\mathcal{S}$ is a surface of constant time in $\mathbb{M}^\#$, since null geodesics in $\mathbb{M}^\#$ are periodic and intersect each such $\mathcal{S}$ in exactly one point (though $\mathcal{S}$ is not, of course, a Cauchy surface). If X and Y are the skies in $\mathbb{PN}$ of any two points in $\mathbb{M}^\#$, then X and Y can be simultaneously deformed to fibres of $\mathbb{PN}$ via an isotopy, so no two skies in $\mathbb{PN}$ are linked.

But now let c be a closed timelike curve in $\mathbb{M}^\#$ which generates the fundamental group, and let x, y and z be any three distinct points on c. Since c is timelike and generates the fundamental group of $\mathbb{M}^\#$, it only intersects the null cone of each of x, y and z at those points respectively.

If we now from identify any of x, y or z as i in $\mathbb{M}^\#$ and remove its null cone, we are left with $\mathbb{M}$: the remaining segment of c is a timelike future pointing curve which approaches i^- to the past and i^+ to the future in $\overline{\mathbb{M}}$, and contains the other two points, which are therefore chronologically separated in this $\mathbb{M}$.

But this says precisely that in $\mathbb{PN}$ with any of X, Y, or Z removed, the other two skies are linked: so we have an analogue of the Borromean rings.

Conversely, suppose that X, Y, Z form a Borromean triple in $\mathbb{PN}$, and suppose (without loss of generality) that $\mathrm{Lk}(X, Y) = 1$ in $\mathbb{PN} \setminus Z$. Then in the copy of $\mathbb{M}$ obtained by removing the null cone of z from $\mathbb{M}^\#$, y is to the future of x and we can choose a future pointing timelike curve which connects x to y, and, if we identify z with i in $\overline{\mathbb{M}}$, extends to i^- in the past and i^+ in the future. Re-inserting the null cone at infinity and identifying, then this curve becomes a future directed timelike curve which generates the fundamental group of $\mathbb{M}^\#$.

We therefore see that there is a causally meaningful analogue to the Borromean triple in the case of skies in $\mathbb{PN}$: three skies in $\mathbb{PN}$ comprise a Borromean triple if and only if they are the skies of three points which lie on a closed timelike curve generating the fundamental group of $\mathbb{M}^\#$.

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